Contributions to combinatorics on words in an abelian context and covering problems in graphs

Élise Vandome

Liège – January 7, 2015
Problem 1

Identifying codes
Guess Who?
Guess Who?

- Individuals: 🌟
- Attributes: 🧐, 🧙, 🧙, 🧢
Guess Who?

- **Individuals:**
  - 📜

- **Attributes:**
  - 🎨, 🕐, 🧐, 🕠
Translation in terms of graphs

- Individuals: \[\text{vertices}\]
- Attributes: \[\text{closed neighbourhoods}\]
Translation in terms of graphs

- **Individuals:** 
  - Vertices

- **Attributes:** 
  - Closed neighbourhoods

How many neighbourhoods/vertices to identify these points?
Translation in terms of graphs

- **Individuals:** 🧑‍👨‍👧‍👦 → vertices

- **Attributes:** 🧑‍👨‍👧‍👦, 🧑, 🧑‍👨, 🧑‍♂️ → closed neighbourhoods

How many neighbourhoods/vertices to identify these points?
Formal definition

An identifying code $C$ is a subset of vertices such that

\begin{itemize}
  \item $\forall u \in V, \quad N[u] \cap C \neq \emptyset$ (domination)
  \item $\forall u \neq v \in V, \quad N[u] \cap C \neq N[v] \cap C$ (separation)
\end{itemize}

[Karpovsky–Chakrabarty–Levitin 1998]

Given a graph $G$, what is the minimum size $\gamma^{ID}(G)$ of an identifying code of $G$?
Formal definition

An identifying code $C$ is a subset of vertices such that

- $\forall u \in V$, $N[u] \cap C \neq \emptyset$ (domination)
- $\forall u \neq v \in V$, $N[u] \cap C \neq N[v] \cap C$ (separation)

[Karpovsky–Chakrabarty–Levitin 1998]

Given a graph $G$, what is the minimum size $\gamma^{ID}(G)$ of an identifying code of $G$?
Formal definition

An identifying code $C$ is a subset of vertices such that

- $\forall u \in V$, $N[u] \cap C \neq \emptyset$ (domination)
- $\forall u \neq v \in V$, $N[u] \cap C \neq N[v] \cap C$ (separation)

[Karpovsky–Chakrabarty–Levitin 1998]

Given a graph $G$, what is the minimum size $\gamma^{ID}(G)$ of an identifying code of $G$?
Formal definition

An identifying code $C$ is a subset of vertices such that

- $\forall u \in V, \quad N[u] \cap C \neq \emptyset$ (domination)
- $\forall u \neq v \in V, \quad N[u] \cap C \neq N[v] \cap C$ (separation)

[Karpovsky–Chakrabarty–Levitin 1998]

Given a graph $G$, what is the minimum size $\gamma^{ID}(G)$ of an identifying code of $G$?
Linear programming formulation

- A variable $x_u$ for each vertex $u$
- Goal: minimize $\sum_{u \in V} x_u$
- Constraints: domination and separation

Minimize \[
\sum_{u \in V} x_u
\]
such that
\[
\sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V \quad \text{(domination)}
\]
\[
\sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V \quad \text{(separation)}
\]
\[
x_u \in \{0, 1\} \quad \forall u \in V
\]
Linear programming formulation

- A variable $x_u$ for each vertex $u$
- Goal: minimize $\sum_{u \in V} x_u$
- Constraints: domination and separation

Minimize

$$\sum_{u \in V} x_u$$

such that

- $\sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V$ (domination)
- $\sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V$ (separation)

$x_u \in \{0,1\} \quad \forall u \in V$

This problem is NP-complete. [Cohen–Honkala–Lobstein–Zémor 2001]
But its fractional relaxation is not!
Linear programming formulation

- A variable $x_u$ for each vertex $u$
- Goal: minimize $\sum_{u \in V} x_u$
- Constraints: domination and separation

Minimize $\sum_{u \in V} x_u$

such that

$\sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V$ (domination)

$\sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V$ (separation)

$x_u \in \{0,1\} \quad [0,1] \quad \forall u \in V$

This problem is NP-complete. [Cohen–Honkala–Lobstein–Zémor 2001]
But its fractional relaxation is not!
Fractional relaxation

Let $\gamma_f^{ID}(G)$ be the optimal solution of the fractional problem.

**Theorem** (Gravier–Parreau–Rottey–Storme–V.)

For any graph $G$, $\gamma_f^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2 \ln |V|) \gamma_f^{ID}(G)$.

Can we compute a closed formula for $\gamma_f^{ID}(G)$?
Fractional relaxation

Let $\gamma_f^{ID}(G)$ be the optimal solution of the fractional problem.

**Theorem** (Gravier–Parreau–Rottey–Storme–V.)

For any graph $G$, $\gamma_f^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2 \ln |V|) \gamma_f^{ID}(G)$.

Can we compute a closed formula for $\gamma_f^{ID}(G)$?

*Yes* in the case of vertex-transitive graphs.
Fractional relaxation

Let $\gamma_f^{ID}(G)$ be the optimal solution of the fractional problem.

**Theorem** (Gravier–Parreau–Rottey–Storme–V.)

For any graph $G$, $\gamma_f^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2 \ln |V|) \gamma_f^{ID}(G)$.

Can we compute a closed formula for $\gamma_f^{ID}(G)$?

Yes in the case of vertex-transitive graphs

A graph is **vertex-transitive** if for any two vertices $u$ and $v$, there is an automorphism sending $u$ to $v$. 
Fractional relaxation

Let $\gamma^f (G)$ be the optimal solution of the fractional problem.

**Theorem** (Gravier–Parreau–Rottey–Storme–V.)

For any graph $G$, $\gamma^f (G) \leq \gamma^D (G) \leq (1 + 2 \ln |V|) \gamma^f (G)$.

Can we compute a closed formula for $\gamma^f (G)$?

Yes in the case of vertex-transitive graphs

A graph is **vertex-transitive** if for any two vertices $u$ and $v$, there is an automorphism sending $u$ to $v$.

Examples:
Fractional relaxation

Let $\gamma_{f}^{ID}(G)$ be the optimal solution of the fractional problem.

**Theorem** (Gravier–Parreau–Rottey–Storme–V.)

For any graph $G$, $\gamma_{f}^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2 \ln |V|) \gamma_{f}^{ID}(G)$.

Can we compute a closed formula for $\gamma_{f}^{ID}(G)$?

Yes in the case of vertex-transitive graphs

A graph is **vertex-transitive** if for any two vertices $u$ and $v$, there is an automorphism sending $u$ to $v$.

Examples:
Fractional relaxation

Let $\gamma_f^{ID}(G)$ be the optimal solution of the fractional problem.

**Theorem** (Gravier–Parreau–Rottey–Storme–V.)

For any graph $G$, $\gamma_f^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2 \ln |V|) \gamma_f^{ID}(G)$.

Can we compute a closed formula for $\gamma_f^{ID}(G)$?

Yes in the case of vertex-transitive graphs

A graph is **vertex-transitive** if for any two vertices $u$ and $v$, there is an automorphism sending $u$ to $v$.

**Properties:**

- All vertices have the same degree.
- There is an optimal solution to the fractional program with all the variables equal.
Fractional value for vertex-transitive graphs

There is an optimal solution with $x_u = \lambda$ for all $u \in V$.

Minimize $\sum_{u \in V} x_u$

such that $\sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V$ (domination)

$\sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V$ (separation)

$x_u \in [0, 1] \quad \forall u \in V$
Fractional value for vertex-transitive graphs

There is an optimal solution with $x_u = \lambda$ for all $u \in V$.

\[
\begin{align*}
\text{Minimize} \quad & \sum_{u \in V} x_u \\
\text{such that} \quad & \sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V \quad \text{(domination)} \\
& \sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V \quad \text{(separation)} \\
& x_u \in [0, 1] \quad \forall u \in V
\end{align*}
\]

where $k$ is the degree of vertices
Fractional value for vertex-transitive graphs

There is an optimal solution with $x_u = \lambda$ for all $u \in V$.

Minimize $\sum_{u \in V} x_u$

such that $\sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V$ (domination)

$\Rightarrow \lambda \geq 1/(k + 1)$

$\sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V$ (separation)

$\Rightarrow \lambda \geq 1/d$

$x_u \in [0, 1] \quad \forall u \in V$

where $k$ is the degree of vertices

$d$ is the smallest size of sets $N[u] \Delta N[v]$
Fractional value for vertex-transitive graphs

There is an optimal solution with \( x_u = \lambda \) for all \( u \in V \).

\[
\text{Minimize} \quad \sum_{u \in V} x_u \\
\text{such that} \quad \sum_{w \in N[u]} x_w \geq 1 \quad \forall u \in V \quad \text{(domination)} \\
\sum_{w \in N[u] \Delta N[v]} x_w \geq 1 \quad \forall u \neq v \in V \quad \text{(separation)} \\
x_u \in [0,1] \quad \forall u \in V
\]

\[
\Rightarrow \lambda \geq \max \left( \frac{1}{k+1}, \frac{1}{d} \right)
\]

where \( k \) is the degree of vertices
\( d \) is the smallest size of sets \( N[u] \Delta N[v] \)
Proposition (Gravier–Parreau–Rottey–Storme–V.)

If $G$ is vertex-transitive, $\gamma_f^D(G) = \lambda \cdot |V| = \frac{|V|}{\min(k+1,d)}$. 
**Proposition** (Gravier–Parreau–Rottey–Storme–V.)

If $G$ is vertex-transitive, $\gamma_f^{ID}(G) = \lambda \cdot |V| = \frac{|V|}{\min(k+1,d)}$.

If $d < k + 1$, the separation condition prevails.

- **power of cycles**
- **Cartesian product of cliques**
If $G$ is vertex-transitive, $\gamma_f^{ID}(G) = \lambda \cdot |V| = \frac{|V|}{\min(k+1,d)}$.

If $k + 1 < d$, the domination condition prevails.
Generalized quadrangles

A generalized quadrangle \( GQ(s, t) \) is an incidence structure of points and lines such that:

- each line contains \( s + 1 \) points,
- each point is on \( t + 1 \) lines,
- if a point \( P \) is not on a line \( L \), there is a unique line through \( P \) intersecting \( L \).

**Adjacency graph**: points are vertices, lines are cliques.
Generalized quadrangles

A generalized quadrangle \( GQ(s, t) \) is an incidence structure of points and lines such that:

- each line contains \( s + 1 \) points,
- each point is on \( t + 1 \) lines,
- if a point \( P \) is not on a line \( L \), there is a unique line through \( P \) intersecting \( L \).

**Adjacency graph:** points are vertices, lines are cliques.

Example: the square grid \( n \times n \) i.e. \( K_n \Box K_n \) is a \( GQ(n - 1, 1) \).
Generalized quadrangles

A generalized quadrangle $GQ(s, t)$ is an incidence structure of points and lines such that:

- each line contains $s + 1$ points,
- each point is on $t + 1$ lines,
- if a point $P$ is not on a line $L$, there is a unique line through $P$ intersecting $L$.

Adjacency graph: points are vertices, lines are cliques.

**Proposition** (Gravier–Parreau–Rottey–Storme–V.)

If $G$ is a vertex-transitive $GQ(s, t)$,

$$
\gamma_f^D(G) = \frac{s^2 t}{st + s + 1} + 1
$$
A generalized quadrangle $\text{GQ}(s, t)$ is an incidence structure of points and lines such that:

- each line contains $s + 1$ points,
- each point is on $t + 1$ lines,
- if a point $P$ is not on a line $L$, there is a unique line through $P$ intersecting $L$.

Adjacency graph: points are vertices, lines are cliques.

**Proposition (Gravier–Parreau–Rottey–Storme–V.)**

If $G$ is a vertex-transitive $\text{GQ}(s, t)$, with $s > 1$, $t > 1$,

$$2^{-5/4} \cdot |V|^{1/4} \leq \gamma_f^{ID}(G) = \frac{s^2 t}{st + s + 1} + 1 \leq 2 \cdot |V|^{2/5}$$
There exists a graph $G$ that is a $GQ(q - 1, q + 1)$ with an identifying code of size $3q$. 

**Proposition** (Gravier–Parreau–Rottey–Storme–V.)
GQ(q − 1, q + 1) with q = 2^\ell

**Proposition** (Gravier–Parreau–Rottey–Storme–V.)

There exists a graph G that is a GQ(q − 1, q + 1) with an identifying code of size 3q.

Projective space on \( \mathbb{F}_q \), \((X_0, X_1, X_2, X_3)\)

- Points: all except \(H_\infty\)
- \(H_\infty\): projective plane \(X_0 = 0\)
GQ(q – 1, q + 1) with q = 2^\ell

**Proposition** (Gravier–Parreau–Rottey–Storme–V.)

There exists a graph G that is a GQ(q – 1, q + 1) with an identifying code of size 3q.

Projective space on \( \mathbb{F}_q \), \((X_0, X_1, X_2, X_3)\)

- **Points:** all except \( H_\infty \)
- **Lines:** the ones through \( C \cup \{N\} \)

\( H_\infty \): projective plane \( X_0 = 0 \)
GQ(q − 1, q + 1) with q = 2^\ell

**Proposition** (Gravier–Parreau–Rottey–Storme–V.)

There exists a graph G that is a GQ(q − 1, q + 1) with an identifying code of size 3q.

---

Projective space on \( \mathbb{F}_q \), \((X_0, X_1, X_2, X_3)\)

- Points: all except \( H_\infty \)
- Lines: the ones through \( C \cup \{N\} \)

Three non coplanar lines through \( N \) form an identifying code.
There exists a graph $G$ that is a $\text{GQ}(q - 1, q + 1)$ with an identifying code of size $3q - 3$.

**Proposition** (Gravier–Parreau–Rottey–Storme–V.)

Projective space on $\mathbb{F}_q$, $(X_0, X_1, X_2, X_3)$

- **Points**: all except $H_\infty$
- **Lines**: the ones through $C \cup \{N\}$

Three non coplanar lines through $N$ form an identifying code.
There exists a graph $G$ that is a $GQ(q - 1, q + 1)$ with an identifying code of size $3q - 3$.

Hence

$$\gamma^{ID}(G) \leq 3q - 3$$
GQ(q − 1, q + 1) with q = 2^\ell

**Proposition** (Gravier–Parreau–Rottey–Storme–V.)

There exists a graph G that is a GQ(q − 1, q + 1) with an identifying code of size 3q − 3.

Hence

\[ \gamma^{ID}(G) \leq 3q − 3 \]

Lower bound?
- Using the fractional value:
  \[ \frac{q^3}{q^2 + q - 1} \leq \gamma^{ID}(G) \]
There exists a graph $G$ that is a $GQ(q - 1, q + 1)$ with an identifying code of size $3q - 3$.

Hence

$$
\gamma^{ID}(G) \leq 3q - 3
$$

Lower bound?

- Using the fractional value:
  $$
  \frac{q^3}{q^2 + q - 1} \leq \gamma^{ID}(G)
  $$

- With discharging methods:
  $$
  3q - 7 \leq \gamma^{ID}(G)
  $$

Finally,

$$
\gamma^{ID}(G) \in \Theta(|V|^{1/3}).
$$
Let $q$ be a prime power.

- There exists a $GQ(q, q)$ with identifying code of size
  \[ 5q \in \Theta(|V|^{1/3}). \]
- There exists a $GQ(q, q^2)$ with identifying code of size
  \[ 5q + 5 \in \Theta(|V|^{1/4}). \]
- There exists a $GQ(q^2, q)$ with identifying code of size
  \[ 5q^2 + 3 \in \Theta(|V|^{2/5}). \]
Proposition (Gravier–Parreau–Rottey–Storme–V.)

For any graph $G$, $\gamma_f^{ID}(G) \leq \gamma^{ID}(G) \leq (1 + 2 \ln |V|) \cdot \gamma_f^{ID}(G)$.

- New families with $\gamma^{ID}$ and $\gamma_f^{ID}$ of the same order $|V|^\alpha$ with $\alpha \in \{1/3, 1/4, 2/5\}$.
- There exists graphs with $\gamma^{ID}$ and $\gamma_f^{ID}$ not of the same order, but $\gamma_f^{ID}$ is constant for them!
- Existence of graphs with $\gamma_f^{ID}$ not constant and $\gamma^{ID}$ not of the same order?
Problem 2

Covering codes
(r, a, b)-covering codes with

- $r$: reach of the emitting stations
- $a$: number of emitting stations within reach of an emitting station
- $b$: number of emitting stations that reach of a phone
Translation in terms of graphs

A set $S \subseteq V$ is an $(r, a, b)$-covering code of $G = (V, E)$ if for any $u \in V$

$$\left| \{ B_r(v) \mid u \in B_r(v), \ v \in S \} \right| = \begin{cases} 
  a & \text{if } u \in S \\
  b & \text{if } u \not\in S.
\end{cases}$$

If $a = 1 = b$, they are called $r$-perfect code. [Biggs 1973]

Finding an $r$-perfect code is NP-complete. [Kratochvíl 1988]
In the case of the infinite square grid $\mathbb{Z}^2$

$r = 3$
In the case of the infinite square grid $\mathbb{Z}^2$

$r = 3$

$a = 3$
In the case of the infinite square grid $\mathbb{Z}^2$

$r = 3$

$a = 3$
In the case of the infinite square grid $\mathbb{Z}^2$

$r = 3$

$a = 3$ and $b = 4$
In the case of the infinite square grid $\mathbb{Z}^2$

$r = 3$

$a = 3$ and $b = 4$
In the case of the infinite square grid $\mathbb{Z}^2$

$r = 3$

$a = 3$ and $b = 4$

$a = 3$ and $b = 4$
Theorem (Puzynina 2008)

For $r \geq 2$, every $(r, a, b)$-covering code of $\mathbb{Z}^2$ is periodic.
Theorem (Puzynina 2008)

For $r \geq 2$, every $(r, a, b)$-covering code of $\mathbb{Z}^2$ is periodic.

Theorem (Axenovich 2003)

If $c$ is an $(r, a, b)$-covering code of $\mathbb{Z}^2$ and $|a - b| > 4$, then $c$ is a $p$-periodic diagonal colouring for some $p = (p, 0)$. 
**Theorem (Puzynina 2008)**

For $r \geq 2$, every $(r, a, b)$-covering code of $\mathbb{Z}^2$ is periodic.

**Theorem (Axenovich 2003)**

If $c$ is an $(r, a, b)$-covering code of $\mathbb{Z}^2$ and $|a - b| > 4$, then $c$ is a $p$-periodic diagonal colouring for some $p = (p, 0)$. 
Projection and folding

Axis

$(0, 0)$

$t = (1, 1)$
Projection and folding

\[ t = (1, 1) \]
Projection and folding

Axis

\( t = (1, 1) \)

Axis

\( p = (6, 0) \)
Projection and folding

Axis

\[ \mathbf{p} = (6, 0) \]

\[ (0, 0) \]

\[ \cdots 0 0 0 0 0 4 3 4 3 4 3 4 0 0 0 0 0 \cdots \]
Projection and folding

Axis

\[ \mathbf{p} = (6, 0) \]
Projection and folding

Axis

\[ p = (6, 0) \]

Axis

\[ p = (6, 0) \]
Projection and folding

\[ \exists a (3, 11, 7)\)-covering code of \( \mathbb{Z}^2 \)
Constant 2-labellings

We only have to study particular colourings in 4 types of cycles!

Type 1 mod, Type 2 mod, Type 3 mod, Type 4 mod
Example of results

**Lemma** (Gravier–V.)

If \( c \) is a non-trivial constant 2-labelling of such cycle, then the number of vertices is a multiple of 3 and \( c \) is 3-periodic of pattern period \( \bullet \bullet \circ \).
Example of results

**Lemma (Gravier–V.)**

If $c$ is a non-trivial constant 2-labelling of such cycle, then the number of vertices is a multiple of 3 and $c$ is 3-periodic of pattern period $\bullet \bullet \circ$.

**Perspectives:**

Many $(1, a, b)$-covering codes of $\mathbb{Z}^d$ are periodic.

[Dorbec–Gravier–Honkala–Mollard 2009]

- Similar periodicity result?
- Which kind of weighted cycles?
Problem 3

Abelian return words
Abelian return words

Classical return word to a factor $u$
Abelian return words

Classical return word to a factor $u$

\[ \text{Abelian equivalence: } \text{silent} \sim_{ab} \text{listen} \quad 00011 \sim_{ab} 10010 \]
Abelian return word to a factor $u$

Abelian equivalence: $\text{silent} \sim_{ab} \text{listen}$ \hspace{1cm} $00011 \sim_{ab} 10010$
Example for the Thue–Morse word

\[ t = 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ \ldots \]

Return words to \( u = 011 \):
Example for the Thue–Morse word

\[ t = 0 \ 1 \ 1 \ 0 \ 1 \ 0 | 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 0 \ 1 \ 1 \ \ldots \]

Return words to \( u = 011 \):

\[ R_{t, u} = \{011010, 011001, 01101001, 0110\} \]
Example for the Thue–Morse word

\[ t = 0 1 1 0 1 0 | 0 1 1 0 0 1 | 0 1 1 0 1 0 0 1 | 0 1 1 0 | 0 1 1 0 1 0 | 0 1 1 \cdots \]

Return words to \( u = 011 \):

\[ \mathcal{R}_{t,u} = \{011010, 011001, 01101001, 0110\} \]

\[ t = 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 1 \cdots \]

Abelian return words to \( u = 011 \):

Example for the Thue–Morse word

\[ t = 011010|011001|01101001|011001|011010|01101001|011010|011 \cdots \]

Return words to \( u = 011 \):

\[ R_{t,u} = \{011010, 011001, 01101001, 0110\} \]

\[ t = 011010|011001|01101001|0110|0110|011010|0110|0110|1101 \cdots \]

Abelian return words to \( u = 011 \):

\[ AR_{t,u} = \{0, 1, 1010, 1100, 10100, 110\} \]
Derived sequence

For the Thue–Morse word \( t = 0110100110010110100101 \cdots \)

Return words to \( u = 011 \): \( R_{t,u} = \{011010, 011001, 01101001, 0110\} \)

\( t = \) |
0 1 1 0 1 0 |
0 1 1 0 0 1 |
0 1 1 0 1 0 0 1 |
0 1 1 0 1 0 1 0 |
0 1 1 0 1 0 |
0 1 1 0 1 0 |
0 1 0 1 1 0 1 0 1 1 \cdots \)
Derived sequence

For the Thue–Morse word $t = 0110100110010110100101 \cdots$

Return words to $u = 011$: $R_{t,u} = \{011010, 011001, 01101001, 0110\}$

$t = |0\ 1\ 1\ 0\ 1\ 0|0\ 1\ 1\ 0\ 0\ 1|0\ 1\ 1\ 0\ 1\ 0\ 0\ 1|0\ 1\ 1\ 0|0\ 1\ 1\ 0\ 1\ 0|0\ 1\ 1\ 1\ 1\ 1\ \cdots$

1 2 3 4 1
Derived sequence

For the Thue–Morse word $t = 0110100110010110100101 \cdots$

Return words to $u = 011$: $R_{t,u} = \{011010, 011001, 01101001, 0110\}$

$t = |0 1 1 0 1 0| 0 1 1 0 0 1 | 0 1 1 0 1 0 0 1 | 0 1 1 0 | 0 1 1 0 1 0 | 0 1 1 \cdots$

\[1 \quad 2 \quad 3 \quad 4 \quad 1\]

Derived sequence: $D_{u}(t) = 1234124312343124123412431 \cdots$
Derived sequence

For the Thue–Morse word $t = 0110100110010110100101 \cdots$

Return words to $u = 011$: $R_{t,u} = \{011010, 011001, 01101001, 0110\}$

$t = |0 1 1 0 1 0|0 1 1 0 0 1|0 1 1 0 1 0 0 1|0 1 1 0|0 1 1 0 1 0|0 1 1 \cdots$

$1 \quad 2 \quad 3 \quad 4 \quad 1$

Derived sequence: $D_{u}(t) = 1234124312343124123412431 \cdots$

Abelian return words to $u = 011$: $AR_{t,u} = \{0, 1, 1010, 1100, 10100, 110\}$

$t = |0 1 1 0 1 0|0 1 1 0 0 1|0 1 1 0 1 0 0 1|0 1 1 0|0 1 1 0 1 0|0 1 1 \cdots$

$1 \quad 2 \quad 3 \quad 1 \quad 4 \quad 2 \quad 1 \quad 2 \quad 5 \quad 2 \quad 1 \quad 6 \quad 1 \quad 2 \quad 3 \quad 1$
Derived sequence

For the Thue–Morse word \( t = 0110100110010110100101 \cdots \)

Return words to \( u = 011 \): \( \mathcal{R}_{t,u} = \{011010, 011001, 01101001, 0110\} \)

\[
\begin{align*}
t &= |011010|011001|01101001|0110|011010|011101101011 \cdots \\
&= 1234124312343124123412431 \cdots
\end{align*}
\]

Derived sequence: \( \mathcal{D}_u(t) = 1234124312343124123412431 \cdots \)

Abelian return words to \( u = 011 \):
\( \mathcal{A}\mathcal{R}_{t,u} = \{0, 1, 1010, 1100, 10100, 110\} \)

\[
\begin{align*}
t &= |011010|011001|01101001|0110|011010|011101101011 \cdots \\
&= 1231421252161231421 \cdots
\end{align*}
\]

Abelian derived sequence: \( \mathcal{E}_u(t) = 1231421252161231421 \cdots \)
**Theorem** (Durand 1998)

A word $w$ is primitive substitutive if and only if the set $\{D_u(w) \mid u \in \text{Pref}(w)\}$ is finite.
A word $w$ is primitive substitutive if and only if the set
$\{D_u(w) \mid u \in \text{Pref}(w)\}$ is finite.

Theorem (Durand 1998)

The Thue–Morse morphism $\sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases}$ is primitive.

$\Rightarrow$ The Thue–Morse word $t$ has \textit{finitely many} derived sequences.
A word \( w \) is primitive substitutive if and only if the set \( \{ \mathcal{D}_u(w) \mid u \in \text{Pref}(w) \} \) is finite.

Theorem (Durand 1998)

The Thue–Morse morphism \( \sigma : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases} \) is primitive.

\( \Rightarrow \) The Thue–Morse word \( t \) has finitely many derived sequences.

Proposition (Rigo–Salimov–V. 2013)

For the Thue–Morse word \( t \), the set \( \{ \mathcal{E}_u(t) \mid u \in \text{Pref}(t) \} \) is infinite.
Characterization of Sturmian words

A word $w$ is Sturmian if for all $n$, the number of factors of length $n$ is $n + 1$.

Example: Fibonacci word

$$f = 0100101001001010010100100101001001 \cdots$$

**Theorem (Vuillon 2001)**

A recurrent infinite word is Sturmian if and only if each of its factor has two return words.
Characterization of Sturmian words

A word \( w \) is Sturmian if for all \( n \), the number of factors of length \( n \) is \( n + 1 \).

Example: Fibonacci word

\[
\mathbf{f} = 0100101001001010010100100101001001010010010100101001001\ldots
\]

**Theorem** (Vuillon 2001)

A recurrent infinite word is Sturmian if and only if each of its factors has two return words.

**Theorem** (Puzynina–Zamboni 2013)

An aperiodic recurrent infinite word is Sturmian if and only if each of its factors has two or three abelian return words.
The set of abelian return words to all prefixes:

\[ \mathcal{APR}_w = \bigcup_{u \in \text{Pref}(w)} \mathcal{APR}_{x,u} \]
The set of abelian return words to all prefixes:

$$\mathcal{APR}_w = \bigcup_{u \in \text{Pref}(w)} \mathcal{APR}_{x,u}$$

**Theorem** (Rigo–Salimov–V. 2013)

Let $w$ be a Sturmian word. The set $\mathcal{APR}_w$ is finite if and only if $w$ does not have a null intercept.
The set of abelian return words to all prefixes:

\[ \mathcal{APR}_w = \bigcup_{u \in \text{Pref}(w)} \mathcal{APR}_{x,u} \]

**Theorem** (Rigo–Salimov–V. 2013)

Let \( w \) be a Sturmian word. The set \( \mathcal{APR}_w \) is finite if and only if \( w \) does not have a null intercept.

\[ f = \text{St}(\alpha, \rho) = 0 \]

where

- \( \alpha = 1/\phi^2 \),
- \( \rho = 1/\phi^2 \),
- \( \phi = (1 + \sqrt{5})/2 \).
The set of abelian return words to all prefixes:

$$\mathcal{APR}_w = \bigcup_{u \in \text{Pref}(w)} \mathcal{APR}_{x,u}$$

**Theorem** (Rigo–Salimov–V. 2013)

Let $w$ be a Sturmian word. The set $\mathcal{APR}_w$ is finite if and only if $w$ does not have a null intercept.

$$f = St(\alpha, \rho) = 01$$

where

- $\alpha = 1/\phi^2$,
- $\rho = 1/\phi^2$,
- $\phi = (1 + \sqrt{5})/2$. 
The set of abelian return words to all prefixes:

\[ \mathcal{APR}_w = \bigcup_{u \in \text{Pref}(w)} \mathcal{APR}_{x,u} \]

**Theorem** (Rigo–Salimov–V. 2013)

Let \( w \) be a Sturmian word. The set \( \mathcal{APR}_w \) is finite if and only if \( w \) does not have a null intercept.

\[
\rho R = R^2 \alpha (\rho) = 010
\]

where
- \( \alpha = 1/\phi^2 \),
- \( \rho = 1/\phi^2 \),
- \( \phi = (1 + \sqrt{5})/2 \).
The set of abelian return words to all prefixes:

\[ \mathcal{APR}_w = \bigcup_{u \in \text{Pref}(w)} \mathcal{APR}_{x,u} \]

**Theorem** (Rigo–Salimov–V. 2013)

Let \( w \) be a Sturmian word. The set \( \mathcal{APR}_w \) is finite if and only if \( w \) does not have a null intercept.

\[ \rho = \frac{1}{\phi^2}, \quad \alpha = \frac{1}{\phi}, \quad \phi = \frac{1 + \sqrt{5}}{2}. \]

\[ f = \text{St}(\alpha, \rho) = 0100 \]
The set of abelian return words to all prefixes:

\[
\mathcal{APR}_w = \bigcup_{u \in \text{Pref}(w)} \mathcal{APR}_{x,u}
\]

**Theorem** (Rigo–Salimov–V. 2013)

Let \( w \) be a Sturmian word. The set \( \mathcal{APR}_w \) is finite if and only if \( w \) does not have a null intercept.

\[\begin{align*}
f &= \text{St}(\alpha, \rho) \\
&= 01001
\end{align*}\]

where

- \( \alpha = 1/\phi^2 \),
- \( \rho = 1/\phi^2 \),
- \( \phi = (1 + \sqrt{5})/2 \).
The set of abelian return words to all prefixes:

\[ \text{APR}_w = \bigcup_{u \in \text{Pref}(w)} \text{APR}_{x,u} \]

**Theorem** (Rigo–Salimov–V. 2013)

Let \( w \) be a Sturmian word. The set \( \text{APR}_w \) is finite if and only if \( w \) does not have a null intercept.

\[ f = \text{St}(\alpha, \rho) = 010010 \]

where
- \( \alpha = 1/\phi^2 \),
- \( \rho = 1/\phi^2 \),
- \( \phi = (1 + \sqrt{5})/2 \).
The set of abelian return words to all prefixes:

\[ \mathcal{APR}_w = \bigcup_{u \in \text{Pref}(w)} \mathcal{APR}_{x,u} \]

**Theorem (Rigo–Salimov–V. 2013)**

Let \( w \) be a Sturmian word. The set \( \mathcal{APR}_w \) is finite if and only if \( w \) does not have a null intercept.

\[ f = \text{St}(\alpha, \rho) = 0100101 \]

where
- \( \alpha = 1/\phi^2 \),
- \( \rho = 1/\phi^2 \),
- \( \phi = (1 + \sqrt{5})/2 \).
The set of abelian return words to all prefixes:

$$\mathcal{APR}_w = \bigcup_{u \in \text{Pref}(w)} \mathcal{APR}_{x,u}$$

**Theorem** (Rigo–Salimov–V. 2013)

Let $w$ be a Sturmian word. The set $\mathcal{APR}_w$ is finite if and only if $w$ does not have a null intercept.

$$f = \text{St}(\alpha, \rho) = 0100101 \cdots$$

where

- $\alpha = 1/\phi^2$,
- $\rho = 1/\phi^2$,
- $\phi = (1 + \sqrt{5})/2$. 

Other result

**Proposition** (Rigo–Salimov–V. 2013)

If \( w \) is an abelian recurrent word such that \( \mathcal{APR}_w \) is finite, then the number of factors of length \( n \) of \( w \) up to abelian equivalence is bounded.

But the converse does not hold in general.
Other result

Proposition (Rigo–Salimov–V. 2013)

If $w$ is an abelian recurrent word such that $\mathcal{APR}_w$ is finite, then the number of factors of length $n$ of $w$ up to abelian equivalence is bounded.

But the converse does not hold in general.

Perspective:

bounded number of factors up to abelian equivalence $+$ condition $\Rightarrow$ abelian recurrent word with finite $\mathcal{APR}_w$?
Problem 4

$\ell$-abelian complexity
Thue–Morse word $t = 0110100110010110\ldots$

Factor complexity $P_t^{(\infty)}$ [Brlek 1989, de Luca–Varricchio 1988]

$$P_t^{(\infty)}(n) = \begin{cases} 
4n - 2 \cdot 2^m - 4 & \text{if } 2 \cdot 2^m < n \leq 3 \cdot 2^m \\
2n + 4 \cdot 2^m - 2 & \text{if } 3 \cdot 2^m < n \leq 4 \cdot 2^m.
\end{cases}$$
Thue–Morse word $t = 0110100110010110 \cdots$

Factor complexity $\mathcal{P}_t^{(\infty)}$ [Brlek 1989, de Luca–Varricchio 1988]

$$\mathcal{P}_t^{(\infty)}(n) = \begin{cases} 
4n - 2 \cdot 2^m - 4 & \text{if } 2 \cdot 2^m < n \leq 3 \cdot 2^m \\
2n + 4 \cdot 2^m - 2 & \text{if } 3 \cdot 2^m < n \leq 4 \cdot 2^m.
\end{cases}$$

Abelian complexity $\mathcal{P}_t^{(1)}$

$$\mathcal{P}_t^{(1)}(2n) = 3 \text{ and } \mathcal{P}_t^{(1)}(2n + 1) = 2$$
\( \ell \)-abelian complexity

Two words \( u, v \) are \( \ell \)-abelian equivalent if

\[ |u|_x = |v|_x \]

for any \( x \) of length at most \( \ell \).

Example:

|   | \( |u|_0 \) | \( |u|_1 \) | \( |u|_{00} \) | \( |u|_{01} \) | \( |u|_{10} \) | \( |u|_{11} \) |
|---|---|---|---|---|---|---|
| 11010011 | 3 | 5 | 1 | 2 | 2 | 2 |
| 11101001 | 3 | 5 | 1 | 2 | 2 | 2 |
Two words $u$, $v$ are $\ell$-abelian equivalent if

$$|u|_x = |v|_x$$

for any $x$ of length at most $\ell$.

Example: 2-abelian equivalent but not 3-abelian equivalent

<table>
<thead>
<tr>
<th>$u$</th>
<th>$u_0$</th>
<th>$u_1$</th>
<th>$u_{00}$</th>
<th>$u_{01}$</th>
<th>$u_{10}$</th>
<th>$u_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11010011</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>11101001</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
**ℓ-abelian complexity**

Two words $u$, $v$ are ℓ-abelian equivalent if

$$|u|_x = |v|_x$$

for any $x$ of length at most $\ell$.

Example: 2-abelian equivalent but not 3-abelian equivalent

<table>
<thead>
<tr>
<th>$u$</th>
<th>$u_0$</th>
<th>$u_1$</th>
<th>$u_{00}$</th>
<th>$u_{01}$</th>
<th>$u_{10}$</th>
<th>$u_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11010011</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>11101001</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Number of factors of length $n$ up to ℓ-abelian equivalence: $P^{(\ell)}_w(n)$

$$P^{(1)}_w(n) \leq \cdots \leq P^{(\ell)}_w(n) \leq P^{(\ell+1)}_w(n) \leq \cdots \leq P^{(\infty)}_w(n)$$

The ℓ-abelian complexity of a word $w$ is the sequence $P^{(\ell)}_w(n)_{n \geq 0}$.

[Karhumäki–Saarela–Zamboni 2013]
2-abelian complexity of the Thue–Morse word
2-abelian complexity of the Thue–Morse word

2-abelian complexity of the Thue–Morse word

- Behaviour? In $\log(n)$ [Karhumäki–Saarela–Zamboni 2014]

Behaviour? In $\log(n)$ [Karhumäki–Saarela–Zamboni 2014]

Regular?
A definition of regularity

A sequence \( s = s(n)_{n \geq 0} \) is \( k \)-regular if the \( \mathbb{Z} \)-module generated by its \( k \)-kernel

\[
\mathcal{K}_k(s) = \{ s(k^en + r)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq r < k^e \}
\]

is finitely generated. [Allouche–Shallit 1992]

Example: 2-kernel of the Thue–Morse word

\[
t = 01101001100101101001011001101001 \cdots
\]
A definition of regularity

A sequence \( s = s(n)_{n \geq 0} \) is \( k \)-regular if the \( \mathbb{Z} \)-module generated by its \( k \)-kernel

\[
\mathcal{K}_k(s) = \{ s(k^e n + r)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq r < k^e \}
\]

is finitely generated. [Allouche–Shallit 1992]

Example: 2-kernel of the Thue–Morse word

\[
t = \ 01101001100101101001011001101001 \cdots
\]

\[
(t_{2n})_{(n \geq 0)} = \]
A definition of regularity

A sequence $s = s(n)_{n \geq 0}$ is \textit{k-regular} if the $\mathbb{Z}$-module generated by its $k$-kernel

$$\mathcal{K}_k(s) = \{s(k^e n + r)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq r < k^e\}$$

is finitely generated. [Allouche–Shallit 1992]

Example: 2-kernel of the Thue–Morse word

$$t = 01101001100101101001011001101001 \ldots$$

$$(t_{2n})_{(n \geq 0)} = 01101001100101101001011001101001 \ldots = t$$
A definition of regularity

A sequence $s = s(n)_{n \geq 0}$ is $k$-regular if the $\mathbb{Z}$-module generated by its $k$-kernel

$$\mathcal{K}_k(s) = \{ s(k^e n + r)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq r < k^e \}$$

is finitely generated. [Allouche–Shallit 1992]

Example: 2-kernel of the Thue–Morse word

$$t = 01101001100101101001011001101001 \ldots$$
$$(t_{2n})_{(n \geq 0)} = 01101001100101101001011001101001 \ldots = t$$
$$(t_{2n+1})_{(n \geq 0)} =$$
A definition of regularity

A sequence $s = s(n)_{n \geq 0}$ is $k$-regular if the $\mathbb{Z}$-module generated by its $k$-kernel

$$\mathcal{K}_k(s) = \{s(k^e n + r)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq r < k^e\}$$

is finitely generated. [Allouche–Shallit 1992]

Example: 2-kernel of the Thue–Morse word

$$t = 01101001100101101001101001 \cdots$$

$$(t_{2n})_{(n \geq 0)} = 01101001100101101001101001 \cdots = t$$

$$(t_{2n+1})_{(n \geq 0)} = 10010110011010010110110100110010110 \cdots = \bar{t}$$
A definition of regularity

A sequence $s = s(n)_{n \geq 0}$ is $k$-regular if the $\mathbb{Z}$-module generated by its $k$-kernel

$$K_k(s) = \{s(k^en + r)_{n \geq 0} : e \geq 0 \text{ and } 0 \leq r < k^e\}$$

is finitely generated. [Allouche–Shallit 1992]

Example: 2-kernel of the Thue–Morse word

$$t = 01101001100101101001011001101001 \cdots$$

$$(t_{2n})_{(n \geq 0)} = 01101001100101101001011001101001 \cdots = t$$

$$(t_{2n+1})_{(n \geq 0)} = 10010110011010010110110100110010110 \cdots = \overline{t}$$

$$K_2(t) = \{t, \overline{t}\}$$
Complexity and regularity

- The factor complexity of a $k$-automatic sequence is $k$-regular. [Carpi–D’Alonzo 2010, Charlier–Rampersad–Shallit 2012]

- The abelian complexity of the Thue-Morse sequence is 2-regular.

- The abelian complexity of the paperfolding sequence is 2-regular. [Madill–Rampersad 2013]

- The abelian complexity of the period-doubling sequence is 2-regular. [Karhumäki–Saarela–Zamboni 2014]
Complexity and regularity

- The factor complexity of a $k$-automatic sequence is $k$-regular. [Carpi–D’Alonzo 2010, Charlier–Rampersad–Shallit 2012]
- The abelian complexity of the Thue-Morse sequence is 2-regular.
- The abelian complexity of the paperfolding sequence is 2-regular. [Madill–Rampersad 2013]
- The abelian complexity of the period-doubling sequence is 2-regular. [Karhumäki–Saarela–Zamboni 2014]

**Question**

Is the $l$-abelian complexity of a $k$-automatic sequence always $k$-regular?
Complexity and regularity

- The factor complexity of a $k$-automatic sequence is $k$-regular. [Carpi–D’Alonzo 2010, Charlier–Rampersad–Shallit 2012]
- The abelian complexity of the Thue–Morse sequence is 2-regular.
- The abelian complexity of the paperfolding sequence is 2-regular. [Madill–Rampersad 2013]
- The abelian complexity of the period-doubling sequence is 2-regular. [Karhumäki–Saarela–Zamboni 2014]

**Question**

Is the $\ell$-abelian complexity of a $k$-automatic sequence always $k$-regular?

Is the 2-abelian complexity of the Thue–Morse word 2-regular?
Regularity via relations

Mathematica experiments

\[ x_{2^e+r} = p_{t}^{(2)}(2^e n + r)_{n \geq 0} \]

\[
\begin{align*}
x_5 &= x_3 \\
x_9 &= x_3 \\
x_{12} &= -x_6 + x_7 + x_{11} \\
x_{13} &= x_7 \\
x_{16} &= x_8 \\
x_{17} &= x_3 \\
x_{18} &= x_{10} \\
x_{20} &= -x_{10} + x_{11} + x_{19} \\
x_{21} &= x_{11} \\
x_{22} &= -x_3 - 2x_6 + x_7 + 3x_{10} + x_{11} - x_{19} \\
x_{23} &= -x_3 - 3x_6 + 2x_7 + 3x_{10} + x_{11} - x_{19} \\
x_{24} &= -x_3 + x_7 + x_{10} \\
x_{25} &= x_7 \\
x_{26} &= -x_3 + x_7 + x_{10} \\
x_{27} &= -2x_3 + x_7 + 3x_{10} - x_{19} \\
x_{28} &= -2x_3 + x_7 + 3x_{10} - x_{14} + x_{15} - x_{19} \\
x_{29} &= x_{15} \\
x_{30} &= -x_3 + 3x_6 - x_7 - x_{10} - x_{11} + x_{15} + x_{19} \\
x_{31} &= -3x_3 + 6x_6 - 2x_{11} - 3x_{14} + 2x_{15} + x_{19} 
\end{align*}
\]
Regularity via relations

Mathematica experiments

\[ x_{2^e+r} = \mathcal{P}_t^{(2)}(2^e n + r)_{n \geq 0} \]

\[
\begin{align*}
x_{32} &= x_8 \\
x_{33} &= x_3 \\
x_{34} &= x_{10} \\
x_{35} &= x_{11} \\
x_{36} &= -x_{10} + x_{11} + x_{19} \\
x_{37} &= x_{19} \\
x_{38} &= -x_3 + x_{10} + x_{19} \\
x_{39} &= -x_{10} + x_{11} + x_{19} \\
x_{40} &= -x_3 + x_{10} + x_{11} \\
x_{41} &= x_{11} \\
x_{42} &= -x_3 + x_{10} + x_{11} \\
x_{43} &= -2x_3 + 3x_{10} \\
x_{44} &= -2x_3 - x_6 + x_7 + 3x_{10} \\
x_{45} &= -x_3 - 3x_6 + 2x_7 + 3x_{10} + x_{11} - x_{19} \\
x_{46} &= -2x_3 - 3x_6 + 2x_7 + 5x_{10} + x_{11} - 2x_{19} \\
x_{47} &= -2x_3 + x_7 + 3x_{10} - x_{19} \\
x_{48} &= -x_3 + x_7 + x_{10} \\
x_{49} &= x_7 \\
x_{50} &= -x_3 + x_7 + x_{10} \\
x_{51} &= -x_3 - 3x_6 + 2x_7 + 3x_{10} + x_{11} - x_{19} \\
x_{52} &= -2x_3 - 3x_6 + 2x_7 + 5x_{10} + x_{11} - 2x_{19} \\
x_{53} &= -2x_3 + x_7 + 3x_{10} - x_{19} \\
x_{54} &= -4x_3 + 3x_6 + x_7 + 3x_{10} - x_{11} - 2x_{14} + x_{15} \\
x_{55} &= -4x_3 + 3x_6 + x_7 + 3x_{10} - x_{11} - 3x_{14} + 2x_{15} \\
x_{56} &= -x_3 + x_{10} + x_{15} \\
x_{57} &= x_{15} \\
x_{58} &= -x_3 + x_{10} + x_{15} \\
x_{59} &= -2x_3 + 3x_6 - x_7 - x_{11} + x_{15} + x_{19} \\
x_{60} &= -4x_3 + 6x_6 + x_{10} - 2x_{11} - 3x_{14} + 2x_{15} + x_{19} \\
x_{61} &= -3x_3 + 6x_6 - 2x_{11} - 3x_{14} + 2x_{15} + x_{19} \\
x_{62} &= -x_3 + 3x_6 - x_7 - x_{10} - x_{11} + x_{15} + x_{19} \\
x_{63} &= x_{15}
\end{align*}
\]
Regularity via relations

If the relations hold, then any sequence $x_n$ for $n \geq 32$ is a linear combination of $x_1, \ldots, x_{19}$. 
Regularity via relations

If the relations hold, then any sequence \(x_n\) for \(n \geq 32\) is a linear combination of \(x_1, \ldots, x_{19}\).

Example: \(x_{154} = P_t^{(2)}(128n + 26)_{n \geq 0}\)

Using \(x_{58} = -x_3 + x_{10} + x_{15}\),

\[
P_t^{(2)}(128n + 26) = P_t^{(2)}(32(4n) + 26) = -P_t^{(2)}(2(4n) + 1) + P_t^{(2)}(8(4n) + 2) + P_t^{(2)}(8(4n) + 7)
\]

\[
= -P_t^{(2)}(8n + 1) + P_t^{(2)}(32n + 2) + P_t^{(2)}(32n + 7).
\]

So

\[
x_{154} = -x_9 + x_{34} + x_{39} = -2x_3 + x_{10} + x_{11} + x_{19}.
\]
Regularity via relations

If the relations hold, then any sequence \( x_n \) for \( n \geq 32 \) is a linear combination of \( x_1, \ldots, x_{19} \).

Example: \( x_{154} = \mathcal{P}_t^{(2)}(128n + 26)_{n \geq 0} \)

Using \( x_{58} = -x_3 + x_{10} + x_{15} \),

\[
\mathcal{P}_t^{(2)}(128n + 26) = \mathcal{P}_t^{(2)}(32(4n) + 26) \\
= -\mathcal{P}_t^{(2)}(2(4n) + 1) + \mathcal{P}_t^{(2)}(8(4n) + 2) + \mathcal{P}_t^{(2)}(8(4n) + 7) \\
= -\mathcal{P}_t^{(2)}(8n + 1) + \mathcal{P}_t^{(2)}(32n + 2) + \mathcal{P}_t^{(2)}(32n + 7).
\]

So

\[
x_{154} = -x_9 + x_{34} + x_{39} = -2x_3 + x_{10} + x_{11} + x_{19}.
\]

**Theorem (Greinecker 2014)**

The relations hold and the 2-abelian complexity is 2-regular.
Symmetry of the form $\mathcal{P}_t^{(2)}(2^{\ell+1} - r) = \mathcal{P}_t^{(2)}(2^\ell + r)$

Some relation between $\mathcal{P}_t^{(2)}(2^\ell + r)$ and $\mathcal{P}_t^{(2)}(r)$
It is the case for many 2-abelian complexity functions.
Symmetry and recurrence relation

**Theorem** (Parreau–Rigo–Rowland–V.)

If $s(n)_{n \geq 0}$ satisfies

$$s(2^{\ell} + r) = \begin{cases} 
  s(r) + c & \text{if } r \leq 2^{\ell-1} \\
  s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1},
\end{cases}$$

then $s(n)_{n \geq 0}$ is 2-regular.

Abelian complexity of the fixed point of $0 \mapsto 12, 1 \mapsto 12, 2 \mapsto 00$
Symmetry and recurrence relation

**Theorem** (Parreau–Rigo–Rowland–V.)

If \( s(n)_{n \geq 0} \) satisfies

\[
s(2^\ell + r) = \begin{cases} 
  s(r) + c & \text{if } r \leq 2^{\ell-1} \\
  s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1},
\end{cases}
\]

then \( s(n)_{n \geq 0} \) is 2-regular.

Abelian complexity of the fixed point of \( 0 \mapsto 12, \ 1 \mapsto 12, \ 2 \mapsto 00 \)

\[
\mathcal{P}_x^{(1)}(2^\ell + r) = \mathcal{P}_x^{(1)}(r) + 3
\]

\[
\mathcal{P}_x^{(1)}(2^{\ell+1} - r) = \mathcal{P}_x^{(1)}(2^\ell + r)
\]
Symmetry and recurrence relation

**Theorem** (Parreau–Rigo–Rowland–V.)

If $s(n)_{n \geq 0}$ satisfies

$$s(2^\ell + r) = \begin{cases} s(r) + c & \text{if } r \leq 2^{\ell-1} \\ s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}, \end{cases}$$

then $s(n)_{n \geq 0}$ is 2-regular.

Abelian complexity of the fixed point of $0 \mapsto 12$, $1 \mapsto 12$, $2 \mapsto 00$

\begin{align*}
\mathcal{P}_x^{(1)}(2^\ell + r) &= \mathcal{P}_x^{(1)}(r) + 3 \\
\mathcal{P}_x^{(1)}(2^{\ell+1} - r) &= \mathcal{P}_x^{(1)}(2^\ell + r)
\end{align*}
Symmetry and recurrence relation

**Theorem** (Parreau–Rigo–Rowland–V.)

If $s(n)_{n \geq 0}$ satisfies

$$s(2^\ell + r) = \begin{cases} 
  s(r) + c & \text{if } r \leq 2^{\ell-1} \\
  s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1},
\end{cases}$$

then $s(n)_{n \geq 0}$ is 2-regular.

Abelian complexity of the fixed point of $0 \mapsto 12$, $1 \mapsto 12$, $2 \mapsto 00$

$$\mathcal{P}_x^{(1)}(2^\ell + r) = \mathcal{P}_x^{(1)}(r) + 3$$

$$\mathcal{P}_x^{(1)}(2^{\ell+1} - r) = \mathcal{P}_x^{(1)}(2^\ell + r)$$
Symmetry and recurrence relation

\textbf{Theorem} (Parreau–Rigo–Rowland–V.)

If $s(n)_{n \geq 0}$ satisfies

$$s(2^\ell + r) = \begin{cases} s(r) + c & \text{if } r \leq 2^{\ell-1} \\ s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}, \end{cases}$$

then $s(n)_{n \geq 0}$ is 2-regular.

Abelian complexity of the fixed point of $0 \mapsto 12, 1 \mapsto 12, 2 \mapsto 00$

$$\mathcal{P}_x^{(1)}(2^\ell + r) = \mathcal{P}_x^{(1)}(r) + 3$$

$$\mathcal{P}_x^{(1)}(2^{\ell+1} - r) = \mathcal{P}_x^{(1)}(2^\ell + r)$$
Symmetry and recurrence relation

**Theorem** (Parreau–Rigo–Rowland–V.)

If \( s(n)_{n \geq 0} \) satisfies

\[
s(2^\ell + r) = \begin{cases} 
  s(r) + c & \text{if } r \leq 2^{\ell-1} \\
  s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1},
\end{cases}
\]

then \( s(n)_{n \geq 0} \) is 2-regular.

Abelian complexity of the fixed point of \( 0 \mapsto 12, \ 1 \mapsto 12, \ 2 \mapsto 00 \)

\[
\mathcal{P}_x^{(1)}(2^\ell + r) = \mathcal{P}_x^{(1)}(r) + 3 \\
\mathcal{P}_x^{(1)}(2^{\ell+1} - r) = \mathcal{P}_x^{(1)}(2^\ell + r)
\]
Symmetry and recurrence relation

**Theorem** (Parreau–Rigo–Rowland–V.)

If $s(n)_{n \geq 0}$ satisfies

$$s(2^\ell + r) = \begin{cases} s(r) + c & \text{if } r \leq 2^{\ell-1} \\ s(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}, \end{cases}$$

then $s(n)_{n \geq 0}$ is 2-regular.

Abelian complexity of the fixed point of $0 \mapsto 12, 1 \mapsto 12, 2 \mapsto 00$

$$\mathcal{P}_x^{(1)}(2^\ell + r) = \mathcal{P}_x^{(1)}(r) + 3$$

$$\mathcal{P}_x^{(1)}(2^{\ell+1} - r) = \mathcal{P}_x^{(1)}(2^\ell + r)$$
How to prove the recurrence and reflection relations

For abelian complexity of the fixed point of $0 \mapsto 12, 1 \mapsto 12, 2 \mapsto 00$

\[ x = 120012121200120012001212120012121200 \ldots \]

- Consider $\Delta_0(n) = \max_{|u|=n} |u|_0 - \min_{|u|=n} |u|_0$.
- It is closely related to the abelian complexity.
- Prove the recurrence and reflection relations for $\Delta_0$. 
How to prove the recurrence and reflection relations

For abelian complexity of the fixed point of $0 \mapsto 12, 1 \mapsto 12, 2 \mapsto 00$

$$x = 1200121212001200120012121200...$$

- Consider $\Delta_0(n) = \max_{|u|=n} |u|_0 - \min_{|u|=n} |u|_0$.
- It is closely related to the abelian complexity.
- Prove the recurrence and reflection relations for $\Delta_0$.

**Theorem** (Parreau–Rigo–Rowland–V.)

- The 2-abelian complexity of the Thue–Morse word is 2-regular.
How to prove the recurrence and reflection relations

For abelian complexity of the fixed point of $0 \mapsto 12$, $1 \mapsto 12$, $2 \mapsto 00$

$$x = 12001212120012001212120012121200...$$

- Consider $\Delta_0(n) = \max_{|u|=n} |u|_0 - \min_{|u|=n} |u|_0$.
- It is closely related to the abelian complexity.
- Prove the recurrence and reflection relations for $\Delta_0$.

**Theorem** (Parreau–Rigo–Rowland–V.)

- The 2-abelian complexity of the Thue–Morse word and of the period-doubling word is 2-regular.
How to prove the recurrence and reflection relations

For abelian complexity of the fixed point of 0 ↦ 12, 1 ↦ 12, 2 ↦ 00

\[ x = 12001212120012001212120012121200... \]

- Consider \( \Delta_0(n) = \max_{|u|=n} |u|_0 - \min_{|u|=n} |u|_0 \).
- It is closely related to the abelian complexity.
- Prove the recurrence and reflection relations for \( \Delta_0 \).

**Theorem (Parreau–Rigo–Rowland–V.)**

- The 2-abelian complexity of the Thue–Morse word and of the period-doubling word is 2-regular.
- The abelian complexity of some other 2-automatic words is 2-regular.
Perspectives

It seems that lots of ($\ell$-)abelian complexity functions satisfy similar recurrence.

For the 3-abelian complexity of period-doubling word $p$, the abelian complexity of the 3-block coding $z$ of $p$ seems to satisfy:

$$P_z^{(1)}(2\ell + r) = \begin{cases} 
P_z^{(1)}(r) + 5 & \text{if } r \leq 2^{\ell-1} \text{ and } r \text{ even} \\
P_z^{(1)}(r) + 7 & \text{if } r \leq 2^{\ell-1} \text{ and } r \text{ odd} \\
P_z^{(1)}(2^{\ell+1} - r) & \text{if } r > 2^{\ell-1}.
\end{cases}$$
Problem 1: Identifying codes

Problem 2: Covering codes

Problem 3: Abelian return words

Problem 4: 2-abelian complexity

Other problem: Syntactic complexity