Chapter III
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke variational principle

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Summary

In this chapter, linear elastic problems in 2D are treated with a new approach of the natural neighbours method (NEM) based on the FRAEIJS de VEUBEKE (FdV) variational principle.

In the spirit of the NEM, the domain is decomposed into $N$ Voronoi cells corresponding to the $N$ nodes distributed inside the domain and on its boundary.

The following discretization hypotheses are admitted:

1. The assumed displacements are interpolated between the nodes with the Laplace interpolation function.
2. The assumed support reactions are constant over each edge $K$ of Voronoi cells on which displacements are imposed.
3. The assumed stresses are constant over each Voronoi cell.
4. The assumed strains are constant over each Voronoi cell.

Introducing these hypotheses in the FdV variational principle produces the set of equations governing the discretized solid.

These equations do not require the calculation of the derivatives of the Laplace interpolation functions and, in the absence of body forces, they only involve numerical integrations on the edges of the Voronoi cells.

These equations are recast in matrix form and it is shown that the discretization parameters associated with the assumptions on the stresses and on the strains can be eliminated at the Voronoi cell level so that the final system of equations only involves the nodal displacements and the assumed support reactions.

These support reactions can be further eliminated from the equation system if the imposed support conditions only involve displacements imposed as constant (in particular displacements imposed to zero) on a part of the solid contour.

Several applications are used to evaluate the method.

A set of patch tests are performed and show that this approach can pass the patch test up to machine precision and that there is no incompressibility locking.

Convergence studies are also made for the case of pure bending of a beam and the numerical solution is compared to the analytical solution of the Theory of Elasticity.

Finally, the case of a square membrane with a hole is also used for convergence evaluation and for comparison with the finite elements solution.

With the present approach, in the absence of body forces, the calculation of integrals over the area of the domain is avoided: only integrations on the edges of the Voronoi cells are required, for which classical Gauss numerical integration with 2 integration points is sufficient to pass the patch test. In addition, the derivatives of the nodal shape functions are not required in the resulting formulation.

The present method also allows solving problems involving nearly incompressible materials without locking.
III.1. Introduction

The present chapter develops a new numerical approach for the solution of 2D linear elastic problems.

The data and unknowns are summarized in figure III.1.

![Diagram of 2D linear elastic problem]

| Data | \( A \): area of the domain in which body forces \( F \) are imposed  
|      | \( S_b \): boundary on which surface tractions \( T \) are imposed  
|      | \( S_u \): boundary on which imposed displacements \( \delta \) are imposed  
|      | \( S = S_b \cup S_u \) with \( N \) the unit outside normal to the domain contour \( S \) |
| Unknowns | \( u \): the displacement field  
|          | \( \varepsilon \): the strain field  
|          | \( \sigma \): the stress field |

Figure III.1. The 2D linear elastic problem

In chapter II, section II.2, the classical approach of the natural neighbours method has been introduced.
The domain contains \( N \) nodes (including nodes on the contour) and the \( N \) Voronoi cells corresponding to these nodes are built.

We have seen that:

- the enforcement of boundary conditions of the type \( u_i = \tilde{u}_i \) on \( S_u \) poses no difficulty if the displacements are interpolated by the Laplace interpolation functions;
- the numerical evaluation of the integrals over the area \( A \) of the domain (or, equivalently on the area of the Voronoi cells) deserves special attention;
- the “stabilized conforming nodal integration” provides an efficient solution for avoiding numerical integration on the Voronoi cells and replaces it by an integral on their contours.

In the present chapter, we will start from the FRAEIJS de VEUBEKE (Fdv) variational principle (chapter II, section II.3) to develop a new approach of the natural neighbours method (NEM).

With this approach, we will see that:

- the derivatives of the Laplace interpolation functions are not necessary
- only numerical integration on the edges of the Voronoi cells are required
- incompressibility locking is avoided

### III.2. Domain decomposition

Since we use the NEM, the domain is decomposed into the \( N \) Voronoi cells corresponding to the \( N \) nodes of the domain, including the nodes on the contour.

The area of the domain is:

\[
A = \sum_{I=1}^{N} A_I \tag{III.1}
\]

with \( A_I \) the area of Voronoi cell \( I \).

We denote \( C_I \) the contour of Voronoi cell \( I \).

The domain contour is the union of some of the edges of the exterior Voronoi cells. These edges are denoted by \( S_K \) and we have

\[
S = \sum_{K=1}^{M} S_K \; ; \quad S_u = \sum_{K=1}^{M_u} S_K \; ; \quad S_t = \sum_{K=1}^{M_t} S_K \; ; \quad S = S_u \cup S_t \Rightarrow M = M_u + M_t \tag{III.2}
\]

where \( M \) is the number of edges composing the contour, \( M_u \) the number of edges on which displacements \( \tilde{u}_i \) are imposed and \( M_t \) the number of edges on which surface tractions \( T_i \) are imposed.
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

We start from the FdV variational principle introduced in chapter II and we recall its expression for completeness (we keep the equation numbers of chapter II).

\[
\delta \Pi = \int_A (\sigma_{ij} - \Sigma_{ij}) \delta \varepsilon_{ij} dA - \int_A \left( \frac{\partial \Sigma_{ji}}{\partial X_j} + F_i \right) \delta u_i dA + \int_A \delta \Sigma_{ij} \left[ \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) - \varepsilon_{ij} \right] dA
\]

\[
+ \int_{S_i} (N_j \Sigma_{ji} - r_i) \delta u_i dS + \int_{S_i} (N_j \Sigma_{ji} - T_i) \delta u_i dS + \int_{S_i} \delta r_i (\tilde{u}_i - u_i) dS = 0
\]

or

\[
\delta \Pi = \delta \Pi_1 + \delta \Pi_2 + \delta \Pi_3 + \delta \Pi_4 + \delta \Pi_5 + \delta \Pi_6 = 0
\]

with the different terms

\[
\delta \Pi_1 = \int A \delta W (\varepsilon_{ij}) dA = \int A \sigma_{ij} \delta \varepsilon_{ij} dA
\]

\[
\delta \Pi_2 = \int A \Sigma_{ij} \left[ \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial X_j} + \frac{\partial \delta u_j}{\partial X_i} \right) - \delta \varepsilon_{ij} \right] dA
\]

\[
\delta \Pi_3 = \int A \delta \Sigma_{ij} \left[ \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) - \varepsilon_{ij} \right] dA
\]

\[
\delta \Pi_4 = - \int A F_i \delta u_i dA
\]

\[
\delta \Pi_5 = - \int_{S_i} T_i \delta u_i dS
\]

\[
\delta \Pi_6 = \int_{S_i} \delta r_i (\tilde{u}_i - u_i) dS - \int_{S_i} r_i \delta u_i dS
\]

For the linear elastic case, the stresses are given by:

\[
\sigma_{ij} = \frac{\partial W (\varepsilon_{ij})}{\partial \varepsilon_{ij}}
\]

and

\[
W (\varepsilon_{ij}) = \frac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl}
\]

where \( C_{ijkl} \) is the classical Hooke’s tensor.

Using the above domain decomposition, these terms become:

\[
\delta \Pi_1 = \sum_{I=1}^N \int_{A_I} \sigma_{ij} \delta \varepsilon_{ij} dA_I
\]

\[
\delta \Pi_2 = \sum_{I=1}^N \sum_{A_I} \left[ \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial X_j} + \frac{\partial \delta u_j}{\partial X_i} \right) - \delta \varepsilon_{ij} \right] dA_I
\]
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$$\delta\Pi_3 = \sum_{I=1}^{N} \int_{A_I} \delta\Sigma_{ij} \left[ \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) - \varepsilon_{ij} \right] dA_I \quad (\text{III. 5})$$

$$\delta\Pi_4 = -\sum_{I=1}^{N} \int_{A_I} F_i \delta u_i dA_I \quad (\text{III. 6})$$

$$\delta\Pi_5 = -\sum_{K=1}^{M_i} \int_{S_K} T_i \delta u_i dS_K \quad (\text{III. 7})$$

$$\delta\Pi_6 = \sum_{K=1}^{M_i} \int_{S_K} \delta r_i (\bar{u}_i - u_i) dS_K - \int_{S_K} r_i \delta u_i dS_K \quad (\text{III. 8})$$

### III.3. Discretization

We make the following discretization hypotheses:

1. The assumed strains $\varepsilon_{ij}$ are constant over each Voronoï cell $I$:
   $$\varepsilon_{ij} = \varepsilon_{ij}' \quad (\text{III. 9})$$

2. The assumed stresses $\Sigma_{ij}$ are constant over each Voronoï cell $I$:
   $$\Sigma_{ij} = \Sigma_{ij}' \quad (\text{III.10})$$

3. The assumed support reactions $r_i$ are constant over each edge $K$ of Voronoï cells on which displacements are imposed:
   $$r_i = r_i^K \quad (\text{III. 11})$$

4. The assumed displacements $u_i$ are interpolated by Laplace interpolation functions:
   $$u_i = \sum_{J=1}^{N} \Phi_{ij} u_i' \quad (\text{III. 12})$$

   where $u_i'$ is the displacement of node $J$ (corresponding to the Voronoï cell $J$).

Some details on the Laplace interpolation functions were given in chapter II, section II.1.2.

As a consequence of (II.34) and (III. 9), the stresses $\sigma_{ij}$ are constant over each Voronoï cell $I$:

$$\sigma_{ij} = \sigma_{ij}' \quad (\text{III. 13})$$

The variations of the independent variables are:

$$\delta\varepsilon_{ij} = \delta\varepsilon_{ij}' \quad (\text{III. 14})$$
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\[
\delta \Sigma_{ij} = \delta \Sigma_{ij}' \quad \text{(III. 15)}
\]

\[
\delta r_i = \delta r_i^K \quad \text{(III. 16)}
\]

\[
\delta u_i = \sum_{I=1}^{N} \Phi_i \delta u_i' \quad \text{(III. 17)}
\]

Introducing these assumptions in (III. 3) to (III. 5), and integrating by parts, we get:

\[
\delta \Pi_i = \sum_{I=1}^{N} \sigma_{ij}' \delta e_{ij}' A_I \quad \text{(III. 18)}
\]

\[
\delta \Pi_2 = \sum_{I=1}^{N} \Sigma_{ij} \int_{A_I} \left[ \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial X_j} + \frac{\partial \delta u_j}{\partial X_i} \right) \right] dA_I - \sum_{I=1}^{N} \Sigma_{ij} \delta e_{ij}' A_I =
\sum_{I=1}^{N} \Sigma_{ij} \int_{C_i} N_j^I \delta u_i \ dC_i - \sum_{I=1}^{N} \Sigma_{ij} \delta e_{ij}' A_I \quad \text{(III. 19)}
\]

\[
\delta \Pi_3 = \sum_{I=1}^{N} \delta \Sigma_{ij} \int_{A_I} \left[ \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial X_j} + \frac{\partial \delta u_j}{\partial X_i} \right) \right] dA_I - \sum_{I=1}^{N} \delta \Sigma_{ij} \delta e_{ij}' A_I =
\sum_{I=1}^{N} \delta \Sigma_{ij} \int_{C_i} N_j^I \delta u_i \ dC_i - \sum_{I=1}^{N} \delta \Sigma_{ij} \delta e_{ij}' A_I \quad \text{(III. 20)}
\]

\[
\delta \Pi_6 = \sum_{K=1}^{M} \left[ \delta r_i^K \int_{S_K} \left( \tilde{u}_i - u_i \right) dS_K - r_i^K \int_{S_K} \delta u_i dS_K \right] \quad \text{(III. 21)}
\]

where \( N_j^I \) is the unit outward normal to the contour of Voronoi cell \( I \).

Introducing in (II.31), we get:

\[
\delta \Pi = \delta \Pi_{IA} + \delta \Pi_{VC} + \delta \Pi_{DC} + \delta \Pi_{EF} = 0 \quad \text{(III. 22)}
\]

with

\[
\delta \Pi_{IA} = \sum_{I=1}^{N} \left( \sigma_{ij}' - \Sigma_{ij} \right) \delta e_{ij}' A_I - \sum_{I=1}^{N} \delta \Sigma_{ij} \delta e_{ij}' A_I \quad \text{(III. 23)}
\]

\[
\delta \Pi_{VC} = \sum_{I=1}^{N} \Sigma_{ij} \int_{C_i} N_j^I \delta u_i \ dC_i + \sum_{I=1}^{N} \delta \Sigma_{ij} \int_{C_i} N_j^I u \ dC_i \quad \text{(III. 24)}
\]

\[
\delta \Pi_{DC} = \sum_{K=1}^{M} \left[ \delta r_i^K \int_{S_K} \left( \tilde{u}_i - u_i \right) dS_K - r_i^K \int_{S_K} \delta u_i dS_K \right] \quad \text{(III. 25)}
\]
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\[
\delta \Pi_{EF} = - \sum_{i=1}^{N} \int_{A_i} F_i \, \delta \mathbf{u}_i \, dA_i - \sum_{K=1}^{M} \int_{S_K} T^K_i \, \delta \mathbf{u}_i \, dS_K
\]  \hspace{1cm} (III. 26)

In (III. 24) to (III. 26), the displacements and virtual displacements are interpolated by (III. 12) and (III. 17) respectively. Substituting in (III. 24), we get:

\[
\delta \Pi_{VC} = \sum_{i=1}^{N} \sum_{j} \left( \int_{S_K} N^K_j \mathbf{u}_i \, dS_K + \int_{S_K} N^K_j \mathbf{u}_i' \, dC_i \right)
\]  \hspace{1cm} (III. 27)

Finally, since the edges of Voronoi cell are straight lines, the outer normal \( N_j \) to edge \( S_K \) is constant along this edge and is denoted \( N^K_j \).

Now, using the discretization (III. 12), we get:

\[
\delta \Pi_{DC} = \sum_{K=1}^{M} \sum_{i} \left( \int_{S_K} \mathbf{u}_i \, dS_K - \int_{S_K} \mathbf{u}_i' \, dS_K \right) - \sum_{K=1}^{M} \sum_{j} \left( \int_{S_K} \mathbf{u}_i' \, dC_i \right)
\]  \hspace{1cm} (III. 28)

Similarly, (III. 26) becomes:

\[
\delta \Pi_{EF} = - \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \int_{S_K} \mathbf{u}_i \, dS_K - \int_{S_K} \mathbf{u}_i' \, dS_K \right) - \sum_{K=1}^{M} \sum_{j} \left( \int_{S_K} \mathbf{u}_i' \, dC_i \right)
\]  \hspace{1cm} (III. 29)

Collecting all the results, we obtain the discretized FdV variational principle.

\[
\delta \Pi = \sum_{i=1}^{N} \left( \sigma^i_{ij} - \Sigma^i_{ij} \right) \delta \epsilon^i_{ij} A_i + \sum_{i=1}^{N} \left( \delta \Sigma^i_{ij} \epsilon^i_{ij} A_i \right)
\]

\[
+ \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \int_{S_K} N^K_j \mathbf{u}_i \, dS_K + \int_{S_K} N^K_j \mathbf{u}_i' \, dC_i \right)
\]

\[
+ \sum_{i=1}^{N} \sum_{j} \left( \int_{S_K} \mathbf{u}_i' \, dC_i \right)
\]

\[
+ \sum_{K=1}^{M} \sum_{j} \left( \int_{S_K} \mathbf{u}_i' \, dC_i \right)
\]
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\[- \sum_{k=1}^{M} t_i^K \left\{ \sum_{j=1}^{N} \delta u_j^I \int_{S_k} \Phi_j dS_K \right\} \]

\[- \sum_{j=1}^{N} \sum_{j=1}^{N} \delta u_j^I \int_{A_j} F_i^I \Phi_j dA_t - \sum_{k=1}^{M} \sum_{j=1}^{N} \delta u_j^I \int_{S_k} T_i^I \Phi_j dS_K = 0 \]  

(III. 30)

### III.4. Equations deduced from the FdV variational principle

Let us reorganize the terms of (III. 30)

\[ \delta \Pi = \sum_{j=1}^{N} \delta e_{ij}^I \left\{ (\sigma_{ij}^I - \Sigma_{ij}^I) A_I \right\} \]

\[ + \sum_{j=1}^{N} \delta \Sigma_{ij}^I \left\{ - e_{ij}^I A_I + \sum_{j=1}^{N} u_j^I A_{ij} \right\} \]

\[ + \sum_{k=1}^{M} \delta t_i^K \left\{ \tilde{U}_i^K - \sum_{j=1}^{N} u_j^I B_{ij} \right\} \]

\[ + \sum_{j=1}^{N} \delta u_j^I \left\{ \sum_{j=1}^{N} (\Sigma_{ij}^I A_{ij} - \tilde{F}_{ij}^I) - \sum_{k=1}^{M} \tilde{T}_i^K \right\} \]

\[ - \sum_{j=1}^{N} \delta u_j^I \left\{ \sum_{k=1}^{M} B_{ij} \right\} = 0 \]  

(III.31)

In this result, the following notations have been used.

\[ A_{ij}^I = \int_{C_i} \gamma_{ij}^I \Phi_j dC_I \]  

(III.32)

\[ B_{ij} = \int_{S_k} \Phi_j dS_K \]  

(III.33)

\[ \tilde{U}_i^K = \int_{S_k} \tilde{u}_i dS_K \]  

(III.34)

\[ \tilde{F}_{ij}^I = \int_{A_j} F_i^I \Phi_j dA_j \]  

(III.35)

\[ \tilde{T}_i^K = \int_{S_k} T_i^I \Phi_j dS_K \]  

(III.36)

Equation (III.32) involves the integration on the contour \( C_i \) of Voronoi cell \( I \).

Equations (III.33) and (III.34) involve the integration on the edge \( S_k \) (belonging to the domain contour) of an exterior Voronoi cell.
We are now able to deduce the Euler equations.

1. In all the Voronoi cells \( I \)
   \[
   \sigma^I_y = \Sigma^I_y \quad \text{for} \quad I = 1, N
   \]  
   (III.37)

   These equations identify the assumed stresses \( \Sigma^I_y \) as the constitutive stresses \( \sigma^I_y \) deduced from the assumed strains \( \varepsilon^I_y \) in each Voronoi cell.

2. In all the Voronoi cells \( I \)
   \[
   \varepsilon^I_y A_I = \sum_{J=1}^{N} u^I_i A^U_J \quad \text{for} \quad I = 1, N
   \]  
   (III.38)

   This is a compatibility equation linking the assumed strains \( \varepsilon^I_y \) in Voronoi cell \( I \) with the assumed nodal displacements \( u^I_i \).

3. On the edges \( K \) of Voronoi cells submitted to imposed displacements
   \[
   \sum_{J=1}^{N} u^I_i B^{KJ} = \tilde{U}^K_i \quad \text{for} \quad K = 1, M_u
   \]  
   (III.39)

   These are also compatibility equations taking account of the imposed displacements \( \tilde{u}_i \) along the domain contour.

4. In all the Voronoi cells \( J \)
   \[
   \sum_{I=1}^{N} (\Sigma^I_y A^U_J - \tilde{F}^U_J) - \sum_{k=1}^{M} \tilde{r}^J_k - \sum_{k=1}^{M} r^K_k B^{KJ} = 0 \quad \text{for} \quad J = 1, N
   \]  
   (III.40)

   These are equilibrium equations taking account of the body forces \( F_i \), the surface tractions \( T_i \) and the assumed support reactions \( r^K_k \).

   We note that, in the developments above, the only term that implies an integration over the area of the Voronoi cells is \( \tilde{F}^U_J = \int_{A_I} F_I \Phi_j dA_I \).

   Hence, if there are no body forces, the problem of choosing integration points is simplified: there are only integrations along the straight edges of Voronoi cells. A classical Gauss integration scheme can be used. Some tests (see section III.6 below) show that 2 integration points give enough precision.
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This property of the present approach eliminates the need for special integration schemes [ATLURI S.N. et al. (1999), ATLURI S.N. and ZHU T. (2000), CUETO E. et al. (2003)] over the area of the domain.

Furthermore, this formulation does not require the derivatives of the shape functions.

So, using the FdV functional as starting point, we obtain the same advantages as with the stabilized conforming nodal integration [CHEN J. S. et al. (2001), YOO J. et al. (2004)].

We also remark that, from the definition of

\[ A_{j}^{IJ} = \frac{1}{C_i} \int_{\Gamma_i} N_j^I \Phi_f dC_i \]

we have

\[ A_{j}^{IJ} = \frac{1}{C_i} \int_{\Gamma_i} N_j^I \Phi_f dC_i = \sum_{all \ K(I)} N_j^{K(I)} \int_{S_{K(I)}} \Phi_f dS_k \]

where \( \sum_{all \ K(I)} \) means the sum over all the edges \( K(I) \) of the Voronoi cell \( I \) and \( N_j^{K(I)} \) is the outward unit normal to that edge.

From this, we see that the calculation of the coefficients \( A_{j}^{IJ} \) and \( B_{j}^{KJ} \) only implies the calculation of integrals of the type \( \int_{S_k} \Phi_f dS_k \) along edges of Voronoi cells.

Finally, in the approach developed here, it is possible to impose displacements \( \tilde{u}_i \) on any edge of any Voronoi cell. From (III.34) and (III.39), it is clear that the imposed displacements are respected in a weighted average sense.

### III.5. Matrix notation

We introduce the following matrix notation.

\[
\{ \varepsilon \}^I = \begin{bmatrix} \varepsilon_{11}^I \\ \varepsilon_{22}^I \\ 2\varepsilon_{12}^I \end{bmatrix}; \quad \{ \sigma \}^I = \begin{bmatrix} \sigma_{11}^I \\ \sigma_{22}^I \\ \sigma_{12}^I \end{bmatrix}; \quad \{ \Sigma \}^I = \begin{bmatrix} \Sigma_{11}^I \\ \Sigma_{22}^I \\ \Sigma_{12}^I \end{bmatrix}; \quad \{ u \}^I = \begin{bmatrix} u_1^I \\ u_2^I \end{bmatrix} \tag{III.41}
\]

\[
\{ \tilde{F} \}^I = \begin{bmatrix} \tilde{F}_1^I \\ \tilde{F}_2^I \end{bmatrix}; \quad \{ \tilde{\tau} \}^I = \begin{bmatrix} \tilde{\tau}_1^I \\ \tilde{\tau}_2^I \end{bmatrix}; \quad \{ \tilde{\tau} \}^I = \sum_{i=1}^{N} \{ \tilde{F} \}^I; \quad \{ \tilde{\tau} \}^I = \sum_{k=1}^{M} \{ \tilde{\tau} \}^I \tag{III.42}
\]

\[
[A]^{IJ} = \begin{bmatrix} A_{11}^{IJ} & 0 \\ 0 & A_{22}^{IJ} \end{bmatrix}; \quad \{ \eta \}^I = \begin{bmatrix} \eta_1^I \\ \eta_2^I \end{bmatrix}; \quad \{ \tilde{\eta} \}^I = \begin{bmatrix} \tilde{\eta}_1^I \\ \tilde{\eta}_2^I \end{bmatrix} \tag{III.43}
\]
Then, we get successively:

\[ \sigma_j^I = \Sigma_j^I \Rightarrow \{\sigma_j^I\} = \{\Sigma_j^I\} \]  

\[ \Sigma_j^I A_j^U \Rightarrow \{A\}^U \{\Sigma_j^I\} ; \quad \tilde{F}_j^U \Rightarrow \{\tilde{F}_j^U\} ; \quad \tilde{T}_j^U \Rightarrow \{\tilde{T}_j^U\} \]  

\[ \sum_{I=1}^{N} \Sigma_j^I A_j^U = \sum_{I=1}^{N} \tilde{F}_j^U + \sum_{K=1}^{M} \tilde{T}_j^K \]  

\[ \Rightarrow \sum_{I=1}^{N} \{A\}^U \{\Sigma_j^I\} - \sum_{K=1}^{M} B^K_j \{r^K_i\} = \{\tilde{F}_j^U\} + \{\tilde{T}_j^K\} \]  

The term \( \sum_{I=1}^{N} \{A\}^U \{\Sigma_j^I\} - \sum_{K=1}^{M} B^K_j \{r^K_i\} \) is the interior nodal force at node \( J \), i.e. in cell \( J \). It is the sum of the contributions \( \{A\}^U \{\Sigma_j^I\} \) of the stresses that are present in all the Voronoi cells \( I \) and of the contributions \( B^K_j \{r^K_i\} \) of the support reactions \( r^K_i \) existing on the contour edges \( K \) where displacements are imposed.

The term \( \{\tilde{F}_j^U\} + \{\tilde{T}_j^K\} \) is the exterior nodal force at node \( J \), i.e. in cell \( J \). It is the sum of:

- the contributions \( \{\tilde{F}_j^U\} \) of the body forces \( F_i \) existing in all the Voronoi cells \( I \)
- the contributions \( \{\tilde{T}_j^K\} \) of the surface tractions \( T_i \) applied on the part \( S_i \) of the domain contour.

Now, consider equation (III.38), it can be written

\[ A_j \{\varepsilon_j^I\} = \sum_{J=1}^{N} \{A\}^{U,J}_j \{u_J^I\} \]  

where \( \{A\}^{U,J}_j \) is the transpose of \( \{A\}^U_j \).

Note that in \( \{\varepsilon_j^I\} \), the third component is \( 2\varepsilon_{12}^I \).

The compatibility equation (III.47) defines the strain \( \{\varepsilon_j^I\} \) in a Voronoi cell \( I \) as the sum of the contributions \( \{A\}^{U,J}_j \{u_J^I\} \) of all the nodes \( J \).

On the edges \( K \) submitted to imposed displacements, we must consider (III.39) that becomes

\[ \sum_{J=1}^{N} B^K_j \{u_J^I\} = \{\tilde{U}_j^K\} \]  

The tables III.1 and III.2 below collect all the results in matrix form.

In these tables, taking account of (III.37), \( \{\Sigma_j^I\} \) is replaced by \( \{\sigma_j^I\} \).
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

Table III. 1. Matrix notations for the linear elastic case

<table>
<thead>
<tr>
<th>Notations and symbols</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( { \varepsilon }^I ) = ( \begin{bmatrix} \varepsilon_{11}^I \ \varepsilon_{22}^I \ 2\varepsilon_{12}^I \end{bmatrix} )</td>
<td>Strains in cell ( I )</td>
</tr>
<tr>
<td>( { \sigma }^I ) = ( \begin{bmatrix} \sigma_{11}^I \ \sigma_{22}^I \ \sigma_{12}^I \end{bmatrix} )</td>
<td>Stresses in cell ( I )</td>
</tr>
<tr>
<td>( { u }^I = \begin{bmatrix} u_1^I \ u_2^I \end{bmatrix} )</td>
<td>Displacements of node ( I ) belonging to cell ( I )</td>
</tr>
<tr>
<td>( { r }^K = \begin{bmatrix} r_1^K \ r_2^K \end{bmatrix} )</td>
<td>Support reactions on edge ( K ) submitted to imposed displacements</td>
</tr>
<tr>
<td>( A_j ); ( C_j )</td>
<td>Area and contour of cell ( I )</td>
</tr>
<tr>
<td>( S_K )</td>
<td>Length of edge ( K ) of a cell</td>
</tr>
<tr>
<td>( \Phi_j )</td>
<td>Interpolant associated with node ( J )</td>
</tr>
</tbody>
</table>

\( \tilde{F}_j^U = \int_{A_j} F_j \Phi_j \, dA_j; \quad \{ \tilde{F} \}^U = \begin{bmatrix} \tilde{F}_1^U \\ \tilde{F}_2^U \end{bmatrix}; \quad \{ F \}^U \) is the nodal force at node \( J \) equivalent to the body forces \( F_i \) applied to the solid |

\( \tilde{T}_i^K = \int_{S_k} T_i \Phi_j \, dS_K; \quad \{ \tilde{T} \}^K = \begin{bmatrix} \tilde{T}_{1}^K \\ \tilde{T}_{2}^K \end{bmatrix}; \quad \{ T \}^K \) is the nodal force at node \( J \) equivalent to the surface tractions \( T_j \) applied to the contour of the solid |

\( \tilde{U}_i^K = \int_{S_k} \tilde{u}_i \, dS_K; \quad \{ \tilde{U} \}^K = \begin{bmatrix} \tilde{U}_2^K \\ \tilde{U}_1^K \end{bmatrix}; \quad \{ U \}^K \) is a generalized displacement taking account of imposed displacements \( \tilde{u}_i \) on edge \( K \) |

\( B_{ij} = \int_{S_k} \Phi_j \, dS_K \) | Integration over the edge \( K \) of a cell |

\( A_{ij} = \int_{C_j} N_j \Phi_j \, dC_I; \quad [A]^U = \begin{bmatrix} A_{ij}^U & 0 \\ 0 & A_{ij}^U \end{bmatrix} \) | \( A_{ij}^U \) can also be computed by \( A_{ij}^U = \sum_{all \, K(j)} N_{ij} B^{K(i)j} \) |
Table III. 2. Discretized equations in matrix form for the linear elastic case

<table>
<thead>
<tr>
<th>Equations</th>
<th>Comments</th>
<th>(III.49)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sum_{I=1}^{N} [A]^{IJ} [\Sigma]^I - \sum_{K=1}^{M} B^{KJ} {r}^K = {\bar{F}}^J + {\bar{t}} )</td>
<td>Equilibrium equation of cell ( J )</td>
<td></td>
</tr>
<tr>
<td>( A_I {c}^I = \sum_{J=1}^{N} [A]^IJ {u}^J )</td>
<td>Compatibility equation for cells ( I )</td>
<td>(III.50)</td>
</tr>
<tr>
<td>( \sum_{J=1}^{N} B^{KJ} {u}^J = {\bar{U}}^K )</td>
<td>Compatibility equation on edge ( K ) submitted to imposed displacements</td>
<td>(III.51)</td>
</tr>
</tbody>
</table>

If we consider a linear elastic material, the constitutive equation for a Voronoi cell \( J \) is

\[ \{\sigma\}^J = [C]^J \{\varepsilon\}^J \tag{III.52} \]

where \([C]^J\) is the Hooke compliance matrix of the elastic material composing cell \( J \).

Introducing (III.50) in (III.52), we get

\[ \{\sigma\}^J = [C]^J \{\varepsilon\}^J = [C^*]^J \sum_{J=1}^{N} [A]^IJ \{u\}^J \tag{III.53} \]

with

\[ [C^*]^J = \frac{1}{A_J} [C]^J \tag{III.54} \]

Then

\[ \sum_{I=1}^{N} [A]^{IJ} [\Sigma]^I = \sum_{L=1}^{N} \sum_{I=1}^{N} \sum_{J=1}^{N} [A]^{IJ} [C^*]^J [A]^{IL} \{u\}^L \tag{III.55} \]

with

\[ [M]^{IL} = \sum_{J=1}^{N} [A]^{IJ} [C^*]^J [A]^{IL} \tag{III.56} \]

Replacing in (III.49), we obtain:

\[ \sum_{L=1}^{N} [M]^{IL} \{u\}^L - \sum_{K=1}^{M} B^{KJ} \{r\}^K = \{\bar{F}\}^J + \{\bar{t}\} \tag{III.57} \]

\[ \sum_{L=1}^{N} [M]^{IL} \{u\}^L = \sum_{K=1}^{M} B^{KJ} \{r\}^K + \{\bar{F}\}^J + \{\bar{t}\} \tag{III.58} \]
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

\[ \sum_{j=1}^{N} B^{kj} \{ u^j \}' = (\tilde{U})^K \] (III.59)

Equations (III.58) and (III.59) constitute an equation system of the form:

\[
\begin{bmatrix}
[M] & \quad -[B]' \\
-B'[T] & 0
\end{bmatrix}
\begin{bmatrix}
\{ q \}' \\
\{ r' \}
\end{bmatrix}
=
\begin{bmatrix}
\{ \tilde{Q} \}' \\
-\{ \tilde{U} \}
\end{bmatrix}
\] (III.60)

with

\[
\begin{align*}
\{ q \} &= \begin{bmatrix} \{ u \}^1 \\ \{ u \}^2 \\ \vdots \\ \{ u \}^N \end{bmatrix} ; \\
\{ \bar{F} \} &= \begin{bmatrix} \{ F \}^1 \\ \{ F \}^2 \\ \vdots \\ \{ F \}^N \end{bmatrix} ; \\
\{ \bar{T} \} &= \begin{bmatrix} \{ T \}^1 \\ \{ T \}^2 \\ \vdots \\ \{ T \}^N \end{bmatrix} ; \\
\{ \tilde{Q} \} &= \{ \bar{F} \} + \{ \bar{T} \}
\end{align*}
\] (III.61)

\[
[M] = \begin{bmatrix}
[M]^{11} & [M]^{12} & \cdots & [M]^{1N} \\
[M]^{21} & [M]^{22} & \cdots & [M]^{2N} \\
\vdots & \vdots & \ddots & \vdots \\
[M]^{N1} & [M]^{N2} & \cdots & [M]^{NN}
\end{bmatrix} ; \\
[B] = \begin{bmatrix}
[B]^{11} & [B]^{12} & \cdots & [B]^{1M_x} \\
[B]^{21} & [B]^{22} & \cdots & [B]^{2M_x} \\
\vdots & \vdots & \ddots & \vdots \\
[B]^{N1} & [B]^{N2} & \cdots & [B]^{NM_x}
\end{bmatrix}
\] (III.62)

It can be easily verified that matrix $[M]$ is symmetric.

Equations (III.51) that, in matrix form, become $[B]' \{ q \} = \{ \tilde{Q} \}$ constitute a set of constraints on the nodal displacements $\{ q \}$.

In particular, if displacements $\tilde{u}_i = 0$ are imposed on the segment $AB$ joining 2 nodes $A$ and $B$ on the domain contour, it is easy to show that (III.51) leads to $u_i^A = 0$ and $u_i^B = 0$.

In such a case, the displacements $u_i^A$ and $u_i^B$ can be removed from the unknowns $\{ q \}$.

This reasoning can be extended to the case of displacements imposed to 0 on any number of similar segments belonging to the contour.

This shows that, despite of the fact that in the initial assumptions (III.9) to (III.12) many discretization parameters appear, most of them are eliminated at the Voronoi cell level, which finally leads to an equation system of the classical form $[M]\{ q \} = \{ \tilde{Q} \}$ with the same characteristics as in the classical approach based on the virtual work principle.
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

### III.6. Numerical integration

For the numerical integration of integrals of the type \( \int_{S_K} \Phi_j dS_k \), Gauss method is used.

Using a local coordinate \(-1 \leq \xi \leq +1\) along edge \(S_K\), such an integral takes the form:

\[
\int_{-1}^{1} f(\xi) d\xi = \sum_{IP=1}^{NP} f(\xi_{IP}) W(IP)
\]  

in which \(IP\) denotes the integration point, \(NP\) the number of integration points, \(\xi_{IP}\) the coordinate of integration point \(IP\) and \(W(IP)\) the weight of integration point \(IP\).

The precision of the scheme has been tested from one to ten integration points. To achieve this, advantage is taken from the fact that the calculation of \(\int_{C_i} A_{ij}^\parallel dS_{ij}^\parallel\) can be performed analytically for a regular distribution of nodes on a square pattern as developed in annex 1.

Table III.3 gives the results for \(A_{1}^\parallel = \int_{C_i} N_1^i \Phi_j dS_{ij}^\parallel\) and \(A_{2}^\parallel = \int_{C_i} N_2^i \Phi_j dS_{ij}^\parallel\).

<table>
<thead>
<tr>
<th>Integration points</th>
<th>(NP=1)</th>
<th>(NP=2)</th>
<th>(NP=3)</th>
<th>(NP=4)</th>
<th>(NP=5)</th>
<th>(NP=6)</th>
<th>(NP=7)</th>
<th>(NP=8)</th>
<th>(NP=9)</th>
<th>(NP=10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative error on (A_{1}^\parallel) (%)</td>
<td>30.13</td>
<td>0.77</td>
<td>4.52</td>
<td>2.62</td>
<td>0.75</td>
<td>0.99</td>
<td>1.72</td>
<td>0.59</td>
<td>0.24</td>
<td>0.57</td>
</tr>
<tr>
<td>Relative error on (A_{2}^\parallel) (%)</td>
<td>12.82</td>
<td>0.32</td>
<td>1.85</td>
<td>1.08</td>
<td>0.31</td>
<td>0.41</td>
<td>0.71</td>
<td>0.24</td>
<td>0.10</td>
<td>0.24</td>
</tr>
</tbody>
</table>

It is seen that the precision does not necessarily increase with the number of integration points. From table III.3, it was decided to use \(NP=2\) in all the subsequent calculations.

### III.7. Applications

#### III.7.1. Patch tests

A set of patch tests in simple tension and in pure shear are performed to validate the method. Unit thickness and plane strain conditions are assumed.
A natural neigbours method for linear elastic problems based on Fraeijs de Veubeke principle.

The domain, the loading, the nodes and the Voronoi cells for the 5 case studies are given in figures III.2 and III.3.

The results are given in tables III.4 and table III.5 for two different Poisson’s ratios: \( \nu = 0 \) and \( \nu = 0.3 \). They are expressed with the help of the following variables.

\[
\text{Average} = \frac{\sum_{K=1}^{N} \sigma^K}{N} ; \quad \text{L2 norm} = \frac{\sum_{K=1}^{N} A_k \sqrt{(\sigma^K_{ij} - \sigma^\text{exact}_{ij})(\sigma^K_{ij} - \sigma^\text{exact}_{ij})} / \sum_{K=1}^{N} A_K}{\sum_{K=1}^{N} \sqrt{\sigma^\text{exact}_{ij} \sigma^\text{exact}_{ij}}} \quad (III.64)
\]

<table>
<thead>
<tr>
<th>Case number and configuration</th>
<th>Loadings and boundary conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: square</td>
<td>( \tilde{u}_y = 0 )</td>
</tr>
<tr>
<td></td>
<td>( \tilde{u}_x = 0 )</td>
</tr>
<tr>
<td>Case 2: rectangular</td>
<td>( \tilde{u}_y = 0 )</td>
</tr>
<tr>
<td></td>
<td>( \tilde{u}_x = \tilde{u}_y = 0 )</td>
</tr>
<tr>
<td>Case 3: rectangular</td>
<td>( \tilde{u}_y = 0 )</td>
</tr>
<tr>
<td>Case 4: square</td>
<td>( t_y = 1000N/mm^2 )</td>
</tr>
<tr>
<td>Case 5: rectangular</td>
<td>( t_y = -1000N/mm^2 )</td>
</tr>
<tr>
<td></td>
<td>( t_x = -1000N/mm^2 )</td>
</tr>
</tbody>
</table>

Figure III.2. Loadings and boundary conditions for the patch tests.
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

<table>
<thead>
<tr>
<th>Case number and number of nodes</th>
<th>Voronoi cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1: 4 nodes</td>
<td></td>
</tr>
<tr>
<td>Case 4: 4 nodes</td>
<td></td>
</tr>
<tr>
<td>Case 2: 45 nodes</td>
<td></td>
</tr>
<tr>
<td>Case 3: 38 nodes</td>
<td></td>
</tr>
<tr>
<td>Case 5: 38 nodes</td>
<td></td>
</tr>
</tbody>
</table>

Figure III.3. Voronoi cells for the patch tests
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

Table III.4. Results of the 5 patch tests with $E = 10^8 \text{N/mm}^2$ and $\nu = 0.3$

<table>
<thead>
<tr>
<th>Case</th>
<th>Configuration</th>
<th>Loading</th>
<th>$N$</th>
<th>$\sigma_{11}$ average ($\text{N/mm}^2$)</th>
<th>$\sigma_{22}$ average ($\text{N/mm}^2$)</th>
<th>$\sigma_{12}$ average ($\text{N/mm}^2$)</th>
<th>L2 norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Square</td>
<td>Simple tension</td>
<td>4</td>
<td>-4.42E-12</td>
<td>1.11E-11</td>
<td>2.87E-16</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Rectangular regular cells</td>
<td>Simple tension</td>
<td>45</td>
<td>2.94E-13</td>
<td>1000</td>
<td>-3.67E-12</td>
<td>5.11E-17</td>
</tr>
<tr>
<td>3</td>
<td>Rectangular irregular cells</td>
<td>Simple tension</td>
<td>38</td>
<td>-7.44E-12</td>
<td>1000</td>
<td>2.16E-11</td>
<td>9.97E-16</td>
</tr>
<tr>
<td>4</td>
<td>Square</td>
<td>Pure shear</td>
<td>4</td>
<td>1.88E-12</td>
<td>-3.99E-17</td>
<td>-1000</td>
<td>3.77E-16</td>
</tr>
<tr>
<td>5</td>
<td>Rectangular irregular cells</td>
<td>Pure shear</td>
<td>38</td>
<td>8.85E-12</td>
<td>7.13E-12</td>
<td>-1000</td>
<td>7.43E-16</td>
</tr>
</tbody>
</table>

Table III.5. Results of the 5 patch tests with $E = 10^8 \text{N/mm}^2$ and $\nu = 0.3$

<table>
<thead>
<tr>
<th>Case</th>
<th>Configuration</th>
<th>Loading</th>
<th>$N$</th>
<th>$\sigma_{11}$ average ($\text{N/mm}^2$)</th>
<th>$\sigma_{22}$ average ($\text{N/mm}^2$)</th>
<th>$\sigma_{12}$ average ($\text{N/mm}^2$)</th>
<th>L2 norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Square</td>
<td>Simple tension</td>
<td>4</td>
<td>-4.42E-12</td>
<td>1.11E-10</td>
<td>2.87E-16</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Rectangular regular cells</td>
<td>Simple tension</td>
<td>45</td>
<td>2.94E-12</td>
<td>1000</td>
<td>-3.67E-12</td>
<td>5.11E-17</td>
</tr>
<tr>
<td>3</td>
<td>Rectangular irregular cells</td>
<td>Simple tension</td>
<td>38</td>
<td>-1.44E-12</td>
<td>1000</td>
<td>5.16E-11</td>
<td>3.10E-16</td>
</tr>
<tr>
<td>4</td>
<td>Square</td>
<td>Pure shear</td>
<td>4</td>
<td>1.92E-12</td>
<td>-1.74E-13</td>
<td>-1000</td>
<td>1.77E-16</td>
</tr>
<tr>
<td>5</td>
<td>Rectangular irregular cells</td>
<td>Pure shear</td>
<td>38</td>
<td>1.37E-12</td>
<td>-4.64E-12</td>
<td>-1000</td>
<td>7.43E-16</td>
</tr>
</tbody>
</table>

These results are computed with 2 integration points on each edge of the Voronoi cells ($NP=2$).

Results with $NP=9$ were also computed but are not significantly different.

It is seen that the patch tests are satisfied up to machine precision.

For case 3, results for a nearly incompressible material with $\nu = 0.49$, $\nu = 0.499$, $\nu = 0.4999$ have also been computed. They are summarized in table III.6 that shows some decrease in the precision of the results.
Table III.6. Results of patch tests (case 3) for a nearly incompressible material

<table>
<thead>
<tr>
<th>( \nu )</th>
<th>L2 norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>3.10E-16</td>
</tr>
<tr>
<td>0.49</td>
<td>5.14E-15</td>
</tr>
<tr>
<td>0.499</td>
<td>1.95E-14</td>
</tr>
<tr>
<td>0.4999</td>
<td>4.70E-13</td>
</tr>
</tbody>
</table>

In fact, this is a problem linked with the deterioration of the conditioning of matrix \([M]\) in (III.60) [WILKINSON, JH. (1965), FRIED I. (1973)].

If there is a small error \([M_e]\) on \([M]\) because of machine precision, the solution is modified and becomes \( \{q\} + \{q_e\} \).

According to [WILKINSON, JH. (1965)]:

\[
\left\| q_e \right\| \leq C \frac{\left\| M_e \right\|}{\left\| M \right\|} \left[ 1 - C \frac{\left\| M_e \right\|}{\left\| M \right\|} \right]
\]

in which \( \left\| \cdot \right\| \) is the L2 norm.

If \( \left\| M_e \right\| \ll \left\| M \right\| \), this result becomes approximately:

\[
\left\| q_e \right\| \leq C \frac{\left\| M_e \right\|}{\left\| M \right\|}
\]

where \( C \) is the condition number of \([M]\).

For a 3D solid discretized in tetrahedral finite elements of size \( h \), it has been shown in [FRIED (1973)] that:

\[
C = c \frac{h^{-2}}{(1+\nu)(1-2\nu)}
\]

where \( c \) is a constant and \( h \) is the size of the elements.

Although this result has not been established in the frame of the natural neighbours method, a simple calculation shows that, when \( \nu \) changes from 0.49 to 0.499 or 0.4999, the error on the solution is multiplied by 10 or 100 or 1000, which explains the results observed in table III.6.

This shows that the patch tests can be passed also for nearly incompressible materials.
III.7.2. Pure bending

Unit thickness and plane strain conditions are assumed.

The loading and boundary conditions are shown on figure III.4.

The left side is fixed horizontally by symmetry.

In order to prevent the rigid body motion in the vertical direction, a small appendix is added to the left side and fixed both in the horizontal and the vertical directions.

Figure III.5 illustrates the Voronoi cells.

\[
\sigma_{\text{max}} = 1000 \text{N/mm}^2
\]

\[
\sigma_{\text{min}} = -1000 \text{N/mm}^2
\]

This is a classical problem of the Theory of Elasticity, the analytical solution of which is known. The strain energy stored in the deformed beam is given by:

\[
W_I = \int_A U_I \, dA
\]

(III.68)

With

\[
M = \int_A \sigma y \, dA = \frac{\sigma_{\text{max}}}{h} \, I \\
I = \int_A y^2 \, dA = \frac{2h^3}{3} \\
\sigma_{xx} = \frac{M}{I} \, Y
\]

(III.69)

the theoretical value of the stored energy becomes:
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

\[ W_{I\_\text{theory}} = \frac{1}{2} \int_A \sigma_{ij} \varepsilon_{ij} dA = \frac{1}{2} \frac{M^2 L(1-v^2)}{EI} \]  

(III.70)

On the other hand, from the numerical calculation, we get

\[ W_{I\_\text{num}} = \sum_{k=1}^N \left\{ \frac{1}{2} \{ \phi^k \}^T \{ \mathbf{K} \} \{ \phi^k \} A_k \right\} \]  

(III.71)

With \( E = 10^5 \text{MPa}, \ v = 0.3 \) and the values of \( \sigma_{\text{max}}, h, L \) (figure III.4), we get:

\[ W_{I\_\text{theory}} = 75.8 J \]

In order to study the influence of the number of nodes and of their distribution, the following procedure has been developed:

a. create a regular pattern of nodes with \((n_X+1)\) nodes in direction \(X\) and \((n_Y+1)\) nodes in direction \(Y\);
   the spacings of the nodes in directions \(X\) and \(Y\) are respectively:
   \[ \Delta_X = L/n_X; \quad \Delta_Y = 2h/n_Y \]

b. move the interior nodes randomly about their previous position by the quantities:
   \[ d_X = \frac{\text{random}}{5} d_X \Delta_X; \quad d_Y = \frac{\text{random}}{5} d_Y \Delta_Y \]
   where \(-0.5 \leq d_r \leq +0.5\) is a uniformly distributed random number and \(0 \leq \text{random} \leq 5\) is a user defined value.

Nodes moving outside the domain are removed.

The results of different calculations are given in table III.7 and summarized in figure III.6 which shows that, with the same \(\text{random}\) value, the results of cases with a more regular Voronoi cell pattern are closer to the theoretical value.

For the case \(\text{random} = 0.1, n_X = 20, n_Y = 8\), figure III.7 shows the stresses \(\sigma_x\) in the different Voronoi cells.

### Table III.7. Energy results for the pure bending test

<table>
<thead>
<tr>
<th>(n_X)</th>
<th>(n_Y)</th>
<th>(N) (nb. of nodes)</th>
<th>(W_{I_\text{num}}) (J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>76</td>
<td>76.9</td>
</tr>
<tr>
<td>15</td>
<td>6</td>
<td>140</td>
<td>76.5</td>
</tr>
<tr>
<td>20</td>
<td>8</td>
<td>244</td>
<td>76.3</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>399</td>
<td>76.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(n_X)</th>
<th>(n_Y)</th>
<th>(N) (nb. of nodes)</th>
<th>(W_{I_\text{num}}) (J)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4</td>
<td>78</td>
<td>80.7</td>
</tr>
<tr>
<td>15</td>
<td>6</td>
<td>140</td>
<td>78.9</td>
</tr>
<tr>
<td>20</td>
<td>8</td>
<td>222</td>
<td>77.8</td>
</tr>
<tr>
<td>20</td>
<td>16</td>
<td>398</td>
<td>76.7</td>
</tr>
</tbody>
</table>
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

To study the effect of near incompressibility, the calculations have also been performed for \( \nu = 0.49, 0.499 \) and 0.4999.

In addition to the convergence on the energy, the convergence on the displacements has also been considered.

To this end, the following norm has been used:
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

\[
L2\text{norm} = \frac{\sum_{i=1}^{N} \sqrt{(u_{i}^{\text{analy}} - u_{i}^{\text{num}})^2 * (u_{i}^{\text{analy}} - u_{i}^{\text{num}})^2}}{f_{\max} * N} \quad (\text{III.72})
\]

where \(u_{i}^{\text{analy}}\) are the nodal displacements computed from the analytical solution while \(u_{i}^{\text{num}}\) are the nodal displacements obtained by the present numerical method.

\[
f_{\max} = \frac{M(1 - \nu^2)}{2EI} L^2 = \frac{\sigma_{\max}(1 - \nu^2)}{2Eh} L^2
\]
is the maximum theoretical displacement of the beam axis.

\(N\) is the number of nodes.

Figures III.8 and III.9 show the convergence curves of the displacements and of the energy for the different values of Poisson’s ratio.

Although there is no formal proof, the results obtained in this section and in the preceding one tend to show that incompressibility locking is avoided in the present formulation.

**Figure III.8.** Displacements convergence for a nearly incompressible material.
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

Chapter III

**Figure III.9.** Energy convergence for a nearly incompressible material.

### III.7.3. Square membrane with a circular hole

The material parameters are:  
\[ E = 200000 \text{N/mm}^2 \text{ and } \nu = 0.3 \]

The geometry and the loading conditions are defined in figure III.10.

Unit thickness and plane strain conditions are assumed.

The strain energy convergence has been studied for different numbers of integration points on the edges of the Voronoi cells (\(NP\)) and different numbers of nodes (\(N\)).

It has been compared with the results of the finite elements method (FEM).

For the finite element analyses, classical 4 nodes isoparametric elements have been used with 4 Gauss integration points for the numerical integrations on the area of the elements. The numbers of nodes used (\(Nel\)) are approximately the same as for the calculations performed with the present approach.

Table III.8 gives the results.
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

The corresponding curves are drawn on figure III.11.

Figure III.10.a. Square membrane with a circular hole: geometry and loading.

Figure III.10.b. Square membrane with a circular hole: studied model.
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

### Table III.8. Strain energy ($J$) for the square membrane with a circular hole

<table>
<thead>
<tr>
<th>$N$</th>
<th>NP=1</th>
<th>NP=2</th>
<th>NP=10</th>
<th>$N_{el}$</th>
<th>NP=4</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>290614</td>
<td>285524</td>
<td>286084</td>
<td>33</td>
<td>238823</td>
</tr>
<tr>
<td>121</td>
<td>284496</td>
<td>282707</td>
<td>282874</td>
<td>119</td>
<td>276975</td>
</tr>
<tr>
<td>441</td>
<td>281741</td>
<td>281070</td>
<td>281146</td>
<td>445</td>
<td>279986</td>
</tr>
<tr>
<td>1681</td>
<td>280715</td>
<td>280517</td>
<td>280542</td>
<td>1645</td>
<td>280100</td>
</tr>
</tbody>
</table>

**Figure III.11**. Square membrane with a circular hole. Strain energy convergence.

It is clearly seen that the present approach converges from above while the finite element method converges from below in the present case. It is also made clear that, for the same number of nodes, the present approach is closer to the converged value.

The convergence of the stress concentration coefficient is also studied. It is defined as:

$$ k = \frac{\sigma_{\text{max}}}{\sigma_{\text{net}}} $$

Where

$$ \sigma_{\text{net}} = \frac{\text{total force in direction } x}{\text{net cross section}} = \frac{200 \times 1000}{320 - 100} = 1500 \text{ N/mm}^2 $$
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

The values are given in tables III.9 and III.10. Figure II.12 illustrates the convergence of $k$.

### Table III.9. Square membrane with a circular hole
Maximum stress near the hole $\sigma_{x,max} \, (N / mm^2)$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$NP=1$</th>
<th>$NP=2$</th>
<th>$NP=10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>3663.72</td>
<td>3577.28</td>
<td>3559.7</td>
</tr>
<tr>
<td>121</td>
<td>3815.36</td>
<td>3749.17</td>
<td>3751.61</td>
</tr>
<tr>
<td>441</td>
<td>3910.19</td>
<td>3891.34</td>
<td>3890.16</td>
</tr>
<tr>
<td>1681</td>
<td>3993.57</td>
<td>3995.68</td>
<td>3993.17</td>
</tr>
</tbody>
</table>

### Table III.10. Square membrane with a circular hole
Stress concentration coefficient

<table>
<thead>
<tr>
<th>$N$</th>
<th>$NP=1$</th>
<th>$NP=2$</th>
<th>$NP=10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td>2.4425</td>
<td>2.3849</td>
<td>2.3731</td>
</tr>
<tr>
<td>121</td>
<td>2.5436</td>
<td>2.4994</td>
<td>2.5011</td>
</tr>
<tr>
<td>441</td>
<td>2.6068</td>
<td>2.5942</td>
<td>2.5934</td>
</tr>
<tr>
<td>1681</td>
<td>2.6624</td>
<td>2.6638</td>
<td>2.6621</td>
</tr>
</tbody>
</table>

**Figure III.12.** Square membrane with a circular hole.
Stress concentration coefficient.
A natural neighbours method for linear elastic problems based on Fraeijs de Veubeke principle.

Finally, figure III.13 illustrates the evolution of the stress distribution with the number of nodes.

<table>
<thead>
<tr>
<th>Nodes</th>
<th>$\sigma_x$</th>
<th>$\sigma_y$</th>
<th>$\sigma_{xy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>36</td>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
</tr>
<tr>
<td>121</td>
<td><img src="image4.png" alt="Image" /></td>
<td><img src="image5.png" alt="Image" /></td>
<td><img src="image6.png" alt="Image" /></td>
</tr>
<tr>
<td>441</td>
<td><img src="image7.png" alt="Image" /></td>
<td><img src="image8.png" alt="Image" /></td>
<td><img src="image9.png" alt="Image" /></td>
</tr>
<tr>
<td>1681</td>
<td><img src="image10.png" alt="Image" /></td>
<td><img src="image11.png" alt="Image" /></td>
<td><img src="image12.png" alt="Image" /></td>
</tr>
</tbody>
</table>

*Figure III.13. Square membrane with a hole: stress distribution*
III.8. Conclusion

The Fraeijs de Veubeke variational principle has been used to develop a natural neighbours method in which the displacements, stresses, strains and surface support reactions can be discretized separately.

It has been shown that the additional degrees of freedom linked with the assumed stresses and strains can be eliminated at the level of the Voronoi cells, finally leading to a system of equations of the same size as in the classical displacement-based method.

With the present approach, in the absence of body forces, the calculation of integrals over the area of the domain is avoided: only integrations on the edges of the Voronoi cells are required, for which classical Gauss numerical integration with 2 integration points is sufficient to pass the patch test. In addition, the derivatives of the nodal shape functions are not required in the resulting formulation.

Hence, the properties of the “stabilized nodal integration method” are recovered using a different approach.

These two methods present a clear advantage over more classical methods using integrations over the area of the domain with the help of a sometimes very high number of integration points.

Concerning the boundary conditions, displacements can be imposed in 2 ways.

- In the spirit of the FdV variational principle, boundary conditions of the type \( u_i = \bar{u}_i \) on \( S_u \) can be imposed in the average sense; hence, any function \( \bar{u}_i = \bar{u}_i(s) \) can be accommodated by the method;
- However, since the natural neighbours method is used, the interpolation of displacements on the solid boundary is linear between 2 adjacent nodes. So, if the imposed displacements \( \bar{u}_i \) are linear between 2 adjacent nodes, they can be imposed exactly. This is obviously the case with \( \bar{u}_i = 0 \). In such a case, it is equivalent to impose the displacements of these 2 adjacent nodes to zero.

On the other hand, the patch tests and the calculations on the bending case tend to show that the present method allows solving problems involving nearly incompressible materials without locking.